Symmetric functions and majorization inequalities: from Newton to Macdonald

Frontiers in Mathematics
Kerala School of Mathematics

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1. Majorization; Newton's inequalities

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- For $n = 2$, $\mathbf{a} = (a_1, a_2) \succcurlyeq \mathbf{b} = (b_1, b_2)$ if $\max(a_1, a_2) \geqslant \max(b_1, b_2)$ and $a_1 + a_2 = b_1 + b_2$.

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- For general n: let $\mathbf{a}^{\downarrow} = (a_1^{\downarrow}, \dots, a_n^{\downarrow})$ be the rearrangement of the a_i in (weakly) decreasing order.
  Now $\mathbf{a} \succcurlyeq \mathbf{b}$ if $\sum_{i=1}^{k} a_i^{\downarrow} \geqslant \sum_{i=1}^{k} b_i^{\downarrow}$ for all $k \leqslant n$, with equality for $k = n$.

If all inequalities hold (but not necessarily the final equality), we say $\mathbf{a}$ *weakly majorizes* $\mathbf{b}$ – denoted $\mathbf{a} \succcurlyeq_w \mathbf{b}$.

Majorization via convex hulls

Proposition (folklore)

Given two real tuples $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, *the following are equivalent.*

1. $\mathbf{a}$ *majorizes* $\mathbf{b}$.
2. $\mathbf{b}$ *belongs to the convex hull of* $\{\sigma(\mathbf{a}) := (a_{\sigma(1)}, \dots, a_{\sigma(n)}) : \sigma \in S_n\}$.

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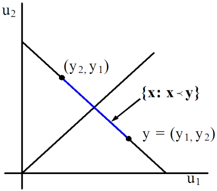
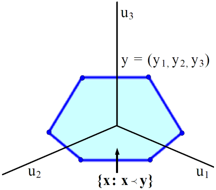
3. $\mathbf{a}$ *weakly majorizes* $\mathbf{b}$ *and* $-\mathbf{a}$ *weakly majorizes* $-\mathbf{b}$.





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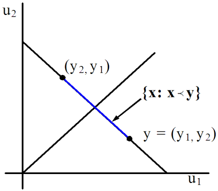

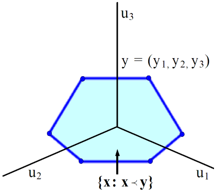

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This led McSwiggen and Novak (2022) to propose + study W-*majorization*, wherein "$\sigma \in S_n$" is replaced by "$\sigma \in W$", a Weyl group of some (other) simple/semisimple Lie type.

Majorization occurs in "nature" – eigenvalues vs. diagonals

Setting where $\sum_i a_i = \sum_i b_i$:
a_i *are the eigenvalues and* b_i *are the diagonal entries.*

- And indeed, majorization does occur among the diagonal entries and eigenvalues of Hermitian matrices. E.g. $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $2, 0$, and $(2, 0) \succcurlyeq (1, 1)$.

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Question: Given two real tuples $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\mathbf{b} = (b_{11}, \dots, b_{nn})$, does there exist a Hermitian matrix $B = (b_{ij})_{n \times n}$ whose eigenvalues are λ_i ?

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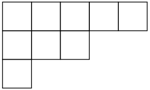
Theorem (Schur (1923)–Horn (1954))

Such a B *exists if and only if* $\boldsymbol{\lambda} \succcurlyeq \mathbf{b}$.

(Special case of: Kostant convexity, Atiyah–Guillemin–Sternberg convexity, Kirwan convexity in symplectic geometry.)

Majorization = dominance order on partitions

A *partition* of $d \geqslant 1$ is a weakly decreasing tuple of nonnegative integers $(\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ such that $\sum_i \lambda_i = d$. One writes $\lambda \vdash d$ and $|\lambda| := d$, and draws



$\longleftrightarrow \quad (5,3,1) \ \vdash \ d = 9.$

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Let $\mathcal{P}_d$ denote the partitions of d.

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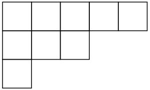

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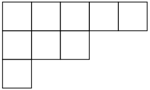

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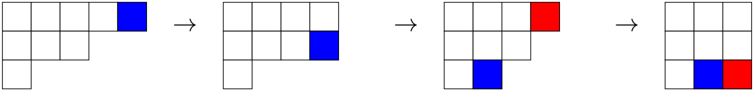
4. *Moreover, $\lambda \succcurlyeq \mu$ in $\mathcal{P}_d$ if and only if μ is obtained from λ by a sequence of "downward cell slidings".*



(5,3,1) (4,4,1) (4,3,2) (3,3,3)

Newton's inequalities

Majorization connects to symmetric functions? E.g. via *Newton's inequalities:*

- Let $p(t) \in \mathbb{R}[t]$ be *negative* real-rooted:

$$p(t) = (t + x_1) \cdots (t + x_n) = \sum_{i=0}^{n} c_i t^{n-i}.$$

- The coefficients c_i are precisely the *elementary symmetric polynomials in* $x := (x_1, \dots, x_n)$*:*

$$c_0 = 1, \quad c_1 = \sum_{i=1}^{n} x_i, \quad c_2 = \sum_{i<j} x_i x_j, \quad \dots, \quad c_n = \prod_{i=1}^{n} x_i.$$

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Theorem (Newton, 1707)

The c_i *satisfy the* Newton inequalities – *i.e. are strongly / ultra log-concave:*

$$\frac{c_i^2}{\binom{n}{i}^2} \geqslant \frac{c_{i+1}}{\binom{n}{i+1}} \frac{c_{i-1}}{\binom{n}{i-1}}.$$

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Quick proof of Newton's inequalities:

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- $c_i = e_i(x_1, \dots, x_n) = e_i(x)$ – elementary symmetric polynomial.
- The number of terms in e_i is precisely $\binom{n}{i}$.
- So Newton's inequalities involve the **normalized** elementary symmetric polynomials:

$$E_i(x)^2 \geqslant E_{i+1}(x) E_{i-1}(x), \quad \text{where} \quad x = (x_1, \dots, x_n) \text{ and } E_i(x) := \frac{e_i(x)}{\binom{n}{i}} = \frac{e_i(x)}{e_i(\mathbf{1})}.$$

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These are in fact (one of the earliest) majorization inequalities!

Elementary symmetric polynomials and majorization

Newton's inequalities say: $(i+1, i-1) \succcurlyeq (i,i)$ and
$E_i(x)E_i(x) \geqslant E_{i+1}(x)E_{i-1}(x)$ on the positive orthant $[0,\infty)^n$ (i.e. $x \geqslant \mathbf{0}$).

More generally, given a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots)$, define the symmetric function

$$E_\lambda(x) := E_{\lambda_1}(x)E_{\lambda_2}(x)\cdots, \qquad x \in [0,\infty)^n.$$

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Theorem (Cuttler–Greene–Skandera, 2011)

Given partitions λ, μ with $|\lambda| = |\mu|$, we have

$$E_\lambda(x) \leqslant E_\mu(x) \quad \forall x \geqslant \mathbf{0} \qquad \textit{if and only if} \qquad \lambda \succcurlyeq \mu.$$

(We will see later: we actually want $\geqslant$, not $\leqslant$! So this is a "reverse" inequality, in a sense – needs to be "fixed".)

2. Majorization for other symmetric functions

Majorization inequalities throughout the ages

This and other majorization inequalities have been shown by:

- Newton (1707) + Cuttler–Greene–Skandera (2011)
- Maclaurin (1729(!))
- Schlömilch (1858)
- Muirhead (1902–03)
- Schur (1920s?)
- Popoviciu (1934)
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We will focus on two more inequalities from the above list.

Monomial symmetric functions

There are other families of symmetric polynomials indexed by partitions λ – e.g., the monomial symmetric functions $m_\lambda(x)$:

$$m_{(3,1)}(x_1, \dots, x_n) = \sum_{i \neq j} x_i^3 x_j^1.$$

- Normalizing this yields $M_\lambda(x) := \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}$.

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- Evaluating on the positive orthant for $\lambda = \lambda_{\max} = (n, 0, \dots, 0)$ which majorizes $\mu = \lambda_{\min} = (1, 1, \dots, 1)$, we get a result known to Pappus of Alexandria (∼400 AD):

Theorem (AM–GM inequality)

For all $n \geqslant 1$, we have

$$M_{(n)}(x) = \frac{x_1^n + \cdots + x_n^n}{n} \quad \geqslant \quad x_1 \cdots x_n = M_{(1,\dots,1)}(x), \qquad \forall x \geqslant \mathbf{0}.$$

Muirhead's inequality

The AM–GM majorization inequality generalizes to all partitions:

Theorem (Muirhead, 1902–03)

Given partitions λ, μ *with* $|\lambda| = |\mu|$, *we have*

$$M_\lambda(x) \geqslant M_\mu(x) \quad \forall x \geqslant \mathbf{0} \qquad \text{if and only if} \qquad \lambda \succcurlyeq \mu.$$

(Note: This inequality is in the "right way" – reverse to Newton.)

Schur polynomials

Schur polynomials defined via Semi-Standard Young Tableaux for λ:

Example 1: Suppose $n = 3$ and $\lambda := (2, 1, 0)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

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$$\begin{aligned} s_{(2,1,0)}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \\ &= (x_1 + x_2)(x_2 + x_3)(x_1 + x_3). \end{aligned}$$

(Weyl Character Formula in type A_2, of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.)

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Example 2: Suppose $n = 3$ and $\mu = (1, 1, 1)$:

1
2
3

Then $s_{(1,1,1)}(x_1, x_2, x_3) = x_1 x_2 x_3$.

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(Normalize these.) Once again, by AM–GM, $S_\lambda(x) \geqslant S_\mu(x)$ for $x \geqslant \mathbf{0}$.

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Theorem (Sra, 2016)

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To summarize:

Upshot: Here are three majorization inequalities:

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Answer: Yes – these are all special cases of *Jack polynomials*.

3. Jack polynomials and majorization

Jack polynomials

- Jack polynomials $p_\lambda^{(\tau)}$ are a family of symmetric functions indexed by a real parameter $\tau > 0$,
  which specialize to $S_\lambda, M_\lambda, E_\lambda$ for specific values of τ.
- Generalize Schur & zonal polynomials ($GL_n(\mathbb{R})$-reps, multivar. statistics).

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- They can be defined recursively (see Wikipedia).
- They can also be defined using their *orthogonality* with respect to the τ-Hall inner product $\langle \cdot, \cdot \rangle_\tau$. When $\tau = 1$, this specializes to a definition of Schur polynomials using the "usual" Hall inner product:

$$\langle s_\lambda, s_\mu \rangle = \langle h_\lambda, m_\mu \rangle = \frac{1}{z_\lambda} \langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu},$$

  where $m_i = \#\{k : \lambda_k = i\}$ and $z_\lambda = \prod_{i \geqslant 1} i^{m_i} m_i!$.

Jack polynomials specialize to s_λ, m_λ, and to...

Special case: 2 variables. Let the partition $\lambda = (a > b \geqslant 0)$.
Under a certain normalization/rescaling,

$$p^{(\tau)}_{(a,b)}(x,y) := \frac{(a-b)!}{(\tau)_{a-b}} \sum_{i=0}^{a-b} \frac{(\tau)_i (\tau)_{a-b-i}}{i!(a-b-i)!} x^{b+i} y^{a-i},$$

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3. What about elementary symmetric polynomials? Take $\tau \to \infty$.

Jack polynomials specialize to "elementary"?

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3. If $\tau \to \infty$, the coefficient goes to $\binom{a-b}{i}$, so

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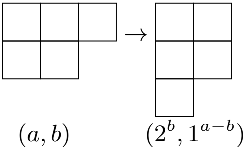

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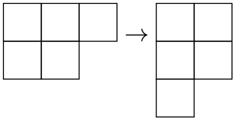

(a,b) $\quad$ $(2^b, 1^{a-b})$

This also fixes the discrepancy! Because:
$\lambda \preccurlyeq \mu \quad \iff \quad \lambda^T \succcurlyeq \mu^T$.

Towards the conjecture

Thus, the three majorization inequalities should be written as:

$$M_\lambda(x) \geqslant M_\mu(x) \ \ \forall x \geqslant \mathbf{0} \qquad \Longleftrightarrow \qquad \lambda \succcurlyeq \mu,$$

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Reformulate using normalized Jack polynomials $P_\lambda^{(\tau)} = \frac{p_\lambda^{(\tau)}(\cdot)}{p_\lambda^{(\tau)}(\mathbf{1})}$:

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(Phrased as "duality" of certain partial orders on partitions vs. among different bases of symmetric polynomials.)

The conjecture and its partial resolution

Conjecture (Chen–K.–Sahi, 2025+)

Let partitions λ, μ be such that $|\lambda| = |\mu|$. The following are equivalent:

1. *For every $0 \leqslant \tau \leqslant \infty$, we have $P_\lambda^{(\tau)}(x) \geqslant P_\mu^{(\tau)}(x) \quad \forall x \geqslant \mathbf{0}$.*
2. *For any fixed $0 \leqslant \tau_0 \leqslant \infty$, we have $P_\lambda^{(\tau_0)}(x) \geqslant P_\mu^{(\tau_0)}(x) \quad \forall x \geqslant \mathbf{0}$.*
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3. *If λ, μ have at most 2 parts (weaker than $n = 2$), then $(3) \implies (1)$.*

Simplest example, in 2 variables

To illustrate our main result, let $n = 2$ and $\lambda = (2,0) \succcurlyeq \mu = (1,1)$. Then

$$p_\lambda^{(\tau)}(x,y) = x^2 + y^2 + \frac{2\tau}{\tau+1}xy, \quad p_\mu^{(\tau)}(x,y) = xy \implies P_\lambda^{(\tau)} - P_\mu^{(\tau)} = \frac{\tau+1}{4\tau+2}(x-y)^2.$$

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Our main result says that • $(3) \implies (4)$ – for all λ, μ; and
• $(3) \implies (1)$ for any number of variables but at most two-part partitions.

4. Weak majorization – and Jack polynomials

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- Many majorization inequalities in the literature, centuries-old even.
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Ingredients of proof: (a) "First-order" approximation of Schur polynomials; (b) Harish-Chandra–Itzykson–Zuber integral; (c) Schur convexity result.

Using weak majorization to characterize majorization

Theorem (K.–Tao, 2021)

Given partitions λ, μ, the following are equivalent:

1. $S_\lambda(x) \geqslant S_\mu(x) \quad \forall x \geqslant \mathbf{0}$.
2. $S_\lambda(x) \geqslant S_\mu(x)$ *for all* $x \in (0,1]^n \cup [1,\infty)^n$.
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  Together, these two are equivalent to: $\lambda \succcurlyeq \mu$. □

Other weak-majorization inequalities

Parallel to Newton – Muirhead – CGS – Sra, the weak-majorization analogues were proved very recently:

Theorem (Chen–Sahi, 2024+)

Given partitions λ, μ, the following are equivalent:

1. $E_{\lambda^T}(x) \geqslant E_{\mu^T}(x) \quad \forall x \geqslant \mathbf{1}$.
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Thus, the analogous conjecture is natural!

The conjecture and its partial resolution

Conjecture (Chen–K.–Sahi, 2025+)

Given partitions λ, μ, the following are equivalent:

1. *For every $0 \leqslant \tau \leqslant \infty$, we have $P_\lambda^{(\tau)}(x) \geqslant P_\mu^{(\tau)}(x) \quad \forall x \geqslant \mathbf{1}$.*
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3. *If λ, μ have at most 2 parts (weaker than $n = 2$), then $(3) \implies (1)$.*

5. Two generalizations:

Macdonald polynomials; W-majorization

Further generalization: Macdonald polynomials

- *Macdonald polynomials* $p_\lambda^{(q,t)}(x)$ are 2-parameter generalization of Schur polynomials – also specialize to Jack polynomials. E.g. for $n = 2$:

$$\begin{aligned} P_{(a,b)}(x,y;q,t) &= P_{(a,b)}(x,y;1/q,1/t) \\ &:= \sum_{i=0}^{a-b} \frac{(q;q)_{a-b}}{(q;q)_i (q;q)_{a-b-i}} \frac{(t;q)_i (t;q)_{a-b-i}}{(t;q)_{a-b}} x^{b+i} y^{a-i} \\ &= \frac{(q;q)_{a-b}}{(t;q)_{a-b}} \sum_{i=0}^{a-b} \frac{(t;q)_i}{(q;q)_i} \frac{(t;q)_{a-b-i}}{(q;q)_{a-b-i}} x^{b+i} y^{a-i}, \end{aligned}$$

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- (Here, $(z;q)_k := \prod_{i=0}^{k-1}(1 - zq^i)$.) For these polynomials, the analogous conjecture can be formulated for majorization… and the same implications hold:

Theorem (Majorization for Macdonald polynomials, Chen–K.–Sahi, 2025+)

1. $(4) \implies (1) \implies (2) \implies (3)$ *always holds.*
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W-majorization

Let $V =$ Euclidean space containing $\Phi =$ crystallographic root system, with Weyl group $W \leqslant O(V)$.
(So W is generated by the reflections in the hyperplanes orthogonal to $\alpha \in \Phi$.)

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Let V = Euclidean space containing Φ = crystallographic root system, with Weyl group $W \leqslant O(V)$.
(So W is generated by the reflections in the hyperplanes orthogonal to $\alpha \in \Phi$.)

Definition *(McSwiggen–Novak):* Given $\lambda, \mu \in V$, say that λ W**-majorizes** μ if μ lies in the convex hull of the orbit $W \cdot \lambda$.

Special case: If Φ is of type A, then $W = S_N$, and then

$$\underline{\lambda \;\; S_N\text{-majorizes } \mu} \quad \text{precisely means} \quad \underline{\lambda \text{ majorizes } \mu}.$$

Riemannian symmetric spaces

- Let $G =$ connected Lie group, $\sigma : G \to G$ an order-2 automorphism. If $K := G^\sigma$ is compact, $X = G/K$ is a *Riemannian symmetric space.*

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- Our work answers: **yes**, at least in type A and for 2 variables.

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