Notes on the geometry of Coxeter groups

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These notes were prepared for a mini-course on the goemetry of Coxeter groups at ‘Frontiers in mathematics’ event at Kerala School of Mathematics, Kozhikode. The goal is to sketch a proof of Moussong’s theorem which states that

Every Coxeter group acts properly discontinuously and cocompactly on a $\mathrm{CAT}(0)$ space.

We will approach Coxeter groups from the lens of geometric group theory and thus these notes do not touch upon on the connection of Coxeter groups with Lie theory.

These notes closely follow **‘Notes on Coxeter groups’ by Chris Cashen**, who in turn used **Mike Davis’ book ‘The geometry of Coxeter groups’**. For a quick introduction to the topic at an undergraduate level, one can refer ‘Office Hours with a Geometric Group Theorist’, from where I borrowed some exercises.

These started as typed notes but very soon evolved into handwritten ones. The notes are organised as follows:

- Introduction to geometric group theory
- Coxeter groups - definitions and preliminaries
- Geometric reflection groups
- Spherical simplices
- Introduction to $\mathrm{CAT}(\kappa)$ geometry
- Construction of Davis complex
- Davis complex is $\mathrm{CAT}(0)$

Intro to Geometric Group theory

The philosophy of geometric group theory is to view

① a group as a metric space.

② and then use the geometry of the space to learn about the group.

Two geometries I am interested in are
- hyperbolic
- CAT(0).

we will see later why groups that act on CAT(0) spaces are 'nice'.

let's focus on ① first.

CAYLEY GRAPH

G be a group

S be a generating set

$\mathrm{Cay}(G,S)$ is a graph with

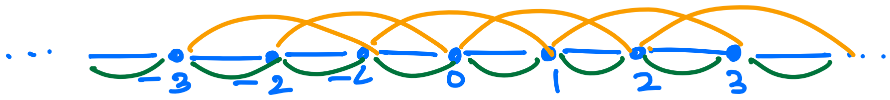
Vertices : elements of G

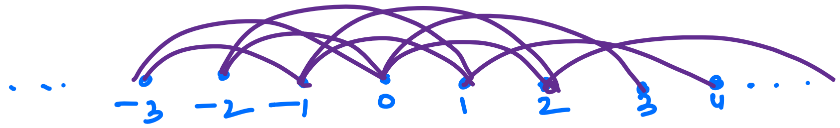
Edges : g —— gs for $s \in S$

metric: each edge has length 1.

Ex. $G = \mathbb{Z}$

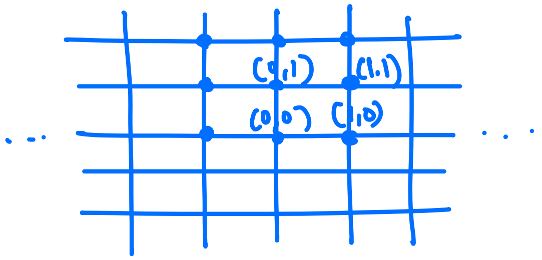
$S = \{1\}$ $S = \{1,2\}$ $S = \{-1,1\}$ $S = \{2,3\}$





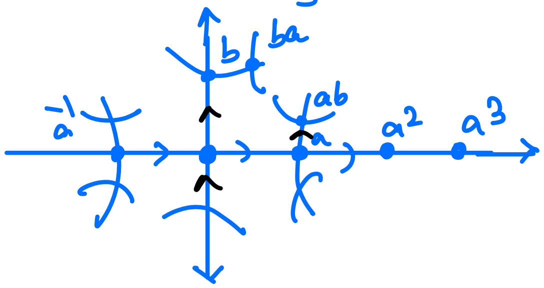
Ex. $G = \mathbb{Z}^2$

$S = \{(1,0),(0,1)\}$



Ex $G = \mathbb{F}_2 = \langle a,b \rangle$

$S = \{a,b\}$



QUASI-ISOMETRY

Def^n: Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A (not necessarily cont.) map $f : X_1 \longrightarrow X_2$ is called a (λ, ε)-quasi-isometric embedding if $\exists$ $\exists$ $\lambda \geq 1$, $\varepsilon \geq 0$ st $\forall$ $x, y \in X_1$

$$\frac{1}{\lambda} d_{X_1}(x,y) - c \leq d_{X_2}(f(x), f(y)) \leq \lambda d_{X_1}(x,y) + c$$

If in addition, $\exists$ const $c \geq 0$ st
$\forall$ $x_2 \in X_2$
$\exists$ $x_1 \in X_1$
st $x_2 \in c$-nbd of $f(x_1)$.

} coarsely surjective.

then f is called a quasi-isometry.

X_1 and X_2 are called quasi-isometric.

Ex. A metric space is QI to a point $\Leftrightarrow$ it has finite diameter.

GGT 1. Let $(\mathbb{Z}, d)$ be the induced metric on $\mathbb{Z}$ from $\mathbb{R}$
then $i : \mathbb{Z} \longrightarrow \mathbb{R}$ is a quasi-isometry.
(isometric emb, coarsely surjective).

GGT 2

Exercise: Every f.g. group is a metric space, well defined upto quasi-isometry.

Pf: let G, S be a finite generating set. Define a metric d_S on G

$d_S(1,g) = |g|_S$ length of shortest pre-image of g under the map $\mathbb{F}(S) \twoheadrightarrow G$ } word length of g wrt S.

$d_S(g,h) = |g^{-1}h|_S$

- act^n of G on itself by left multiplication gives an embedding $G \longrightarrow \mathrm{Isom}(G, d_S)$
- act^n by right multiplicat^n by γ given an isom only if $\gamma \in \mathrm{Center}(G)$.

Let S' be another finite gen set, with metric $d_{S'}$

Claim: $\exists\, \lambda$ st

$$\frac{1}{\lambda} d_{S'}(g,h) \le d_S(g,h) \le \lambda\, d_{S'}(g,h)$$

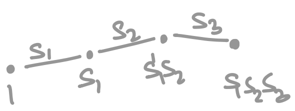
Pf: let $s_i \in S$

let $d_{S'}(1, s_i) = \lambda_i$

let $\lambda = \max \lambda_i$

$$\text{then } d_{S'}(1,g) = d_{S'}(1, \overbrace{s_1 s_2 s_3 \cdots s_n}^{\text{shortest representative.}})$$
$$\le d_{S'}(1,s_1) + d_{S'}(s_1, s_1 s_2) + \dots + d_{S'}(s_1 \cdots s_{n-1}, s_1 \cdots s_n)$$
$$= d_{S'}(1,s_1) + d_{S'}(1,s_2) + \dots + d_{S'}(1,s_n)$$
$$\le n\lambda.$$

$\underline{n} \hookrightarrow |g|_S = d_S(1,g)$



④ $Cay(G,S)$ metric graph with edge lengths 1.
The induced metric on the vertex set is exactly d_S.

∴ $(G, d_S) \xhookrightarrow{i} Cay(G,S)$
is a Quasi-isometry.

$\Rightarrow$ $Cay(G,S) \underset{QI}{\simeq} Cay(G,S')$.

" No unique Cayley graph, but they all look about the same if you squint your eyes "

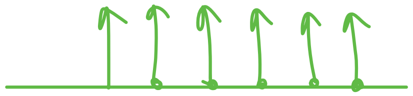
Ex. $\mathbb{Z}$, S, S'

→ **Quasi-isometries arising from group actions:**
(Milnor-Švarc lemma).

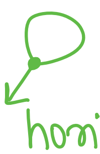
Def^n: $G \curvearrowright X$ space

(1a) The action is called cocompact if the quotient space X/G (with quotient top) is compact.

Ex. $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$
$\mathbb{Z} \curvearrowright \mathbb{R}$
$\pi_1(S_g) \curvearrowright \mathbb{H}^2$.

Ex. 

$\mathbb{Z} \curvearrowright X$ by translation

$X/\mathbb{Z}$ =  not compact.

Ex. $\mathbb{Z} \curvearrowright \mathbb{R}^2$ by horizontal transl.

Better def^n:

(1b) $G \curvearrowright X$ is cocompact if there exists a compact set $K \subseteq X$ such that $X = G \cdot K$

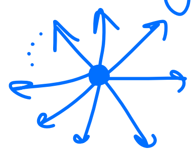
② $G \curvearrowright X$ is free if $\forall\, x \in X$, $g \in G \setminus \{1\}$, $gx \neq x$.

③ $G \curvearrowright (X,d)$ by isometries if $d(gx, gy) = d(x,y)$.

(4b) X top space
G acts properly on X
if for every compact set K in X
$\#\{ g \in G \mid K \cap gK \neq \emptyset \}$ is finite.

Ex: X metric space
$\tilde{X}$ gets an induced metric.
$\pi_1(X) \curvearrowright \tilde{X}$ by deck transf
acts properly.

Ex: not acting properly:



$\circlearrowleft \mathbb{Z}$
by rotation

GGT3

Let X be a CW complex, which is locally finite and finite dim'l.
Suppose a f.g. group G acts on X combinatorially.
(sends n-cells to n-cells)

Suppose cell stabilizers are finite.
Can you conclude that $G \curvearrowright X$ is properly discontinuous?

Lemma: (X,d) metric space

G group with finite gen set S and word metric d_S.

If $G \curvearrowright X$ by isometries,

then $\forall\, x_0 \in X$,

$\exists\, \mu > 0$

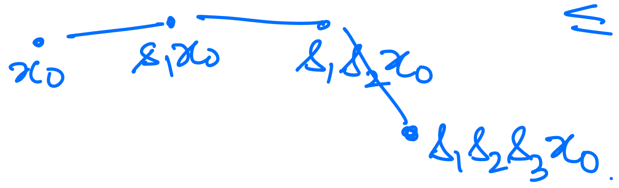
s.t. $d_X(gx_0, g'x_0) \leq \mu\, d_S(g, g')$ for all $g, g' \in G$

Proof: let $\mu = \max\{ d(x_0, sx_0) \mid s \in S \cup S^{-1} \}$

if $d_S(g, g') = n$ then $g^{-1}g' = s_1 \cdots\cdots s_n$
for some $s_1, \ldots\ldots s_n \in S \cup S^{-1}$

let $g_i = s_1 \cdots\cdots s_i$

$$\begin{aligned} d(gx_0, g'x_0) &= d(x_0, g^{-1}g'x_0) \\ &\leq d(x_0, s_1x_0) + d(s_1x_0, s_1s_2x_0) + \cdots\cdots \\ &= \sum d(x_0, s_ix_0) \\ &\leq \mu n = \mu\, d_S(g, g') \end{aligned}$$



∎

Lemma: X top space

$G \curvearrowright X$ by homeomorphisms.

let $U \subset X$ be an openset such that $G \cdot U = X$

If X is connected, then $S = \{ g \in G \mid g \cdot U \cap U \neq \emptyset \}$ generates G.

proof:

let $H \subseteq G$ generated by S.

let $Y = H \cdot U$

$Y' = (G \setminus H) \cdot U$

If $Y \cap Y' \neq \emptyset$

then $\exists\, h \in H,\ g \in G \setminus H$ st $hU \cap gU \neq 0$

$U \cap \bar{h} g U \neq \emptyset$

$\Rightarrow \bar{h} g \in S$

$\Rightarrow g \in HS = H \quad \rightarrow\leftarrow$

$\therefore\ X = Y \sqcup Y'$ disjoint opensets

Y non-empty

X connected

$\Rightarrow Y' = \emptyset$ and $X = Y = H \cdot U = G \cdot U$

$\Rightarrow S$ generates G.

Recall, $G = PSL(2, \mathbb{Z}) \curvearrowright \mathbb{H}^2$ by isom

$U = \varepsilon$-nhd of fundam domain F.

$S = \{ T, \bar{T}, R \}$

Then S generates $G = PSL(2, \mathbb{Z})$.

earlier we used the fact that F was a fundamental domain.

Remark: If G acts cocompactly on (X,d)
then $\exists$ compact set K st $G\cdot K = X$
choose a metric ball $B(x_0,r)$ containing K
and set $U = B(x_0,r)$

Now if in addition, the action is proper, then
the set S is finite.

Corollary: $G \curvearrowright (X,d)$ metric space
properly and compactly by isometries
then G is finitely generated

in fact ($\Leftrightarrow$ f.p).

Proposition (Milnor–Švarc-lemma)
(Fundamental theorem of GGT).

X proper geod metric space (length space)
$G \curvearrowright X$ properly, cocomp by isom
Then Γ is f.g. and for any $x_0 \in X$
the orbit map
$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g\cdot x_0 \end{aligned}$$
is a quasi-isometry.

Proof: let K be a compact set st $G \cdot K = X$

choose x_0 and $D>0$ st $K \subset B(x_0, D/3)$

$S = \{ g \in G \mid g B(x_0, D) \cap B(x_0, D) \neq \phi \}$

proper action $\Rightarrow$ S is finite

previous lemma $\Rightarrow$ S generates G.

d_S word metric

$\exists\, \mu > 0$ st. $d_X(g x_0, g' x_0) \leq \mu\, d_S(g, g')$

one side of the ineq.

need to show: $d_S(1, g) \leq c_1 d_X(x_0, g x_0) + c_2$.

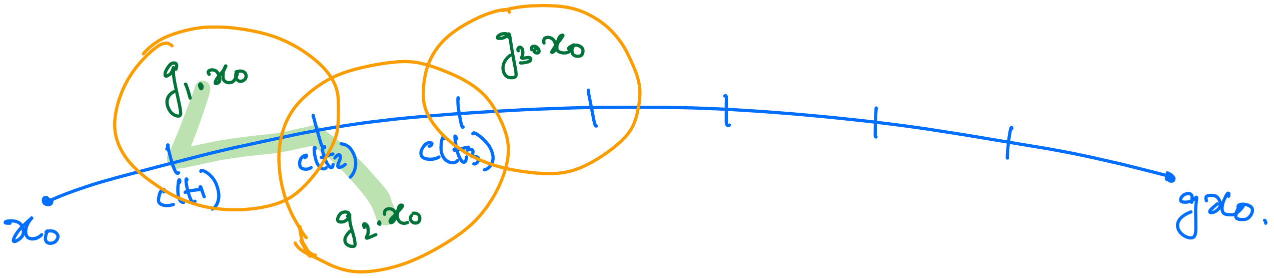
for $g \in G$

consider a ~~path~~ geod joining x_0 and $g x_0$ in X

$c : [0,1] \longrightarrow X$ of finite length.

Partition $[0,1]$ st $d(c(t_i), c(t_{i+1})) \leq D/3$

$0 = t_0 < t_1 < \dots < t_n = 1$



for each i, choose g_i such that $d(c(t_i), g_i x_0) \leq D/3$.

(By cocompactness)

$g_0 = 1$, $g_n = g$

$n \frac{D}{3}$

Then $d(g_{i-1} x_0, g_i x_0) \leq D$.

$\Rightarrow g_{i-1}^{-1} g_i \in S \quad := s_i$

Another proof that S generates G.

$$g = \boxed{g_0}\underbrace{(\bar{g}_0 g_1)}_{s_1}\underbrace{(\bar{g}_1 g_2)}_{s_2}\cdots\cdots\underbrace{(\bar{g}_{n-1}\boxed{g_n})}_{s_n}$$

$g_0 \to 1$, $g_n \to g$.

$g = g_n$.

$$= s_1 s_2 \ldots s_n.$$

(X length space
choose c to have length less than $d(x_0, gx_0) + 1$)

c geod ⇒ length of $c = d(x_0, gx_0)$

choose the coarsest partition possible

↳ last pc could be smaller.

$$(n-1)\frac{D}{3} \leq d(x_0, gx_0) \leq n \cdot \frac{D}{3}$$

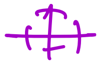
also $|g|_s \leq n$.

$$|g|_s \leq n \leq \frac{3}{D} d(x_0, gx_0) + 1$$ ∎

Ex. $\pi_1(S_g) \underset{QI}{\simeq} \pi_1(S_h)$ $g, h > 2$.

Any two closed hyp 3-mfld groups are QI.

Ex. $\mathbb{Z}^m \underset{QI}{\simeq} \mathbb{Z}^n \iff m = n.$

GGT4 Ex. $\mathbb{F}_2 \underset{QI}{\simeq} \mathbb{F}_3$ $\mathbb{F}_3 \curvearrowright$  properly, cocompactly.

GGT5 Ex. $G \overset{\bullet}{<} G'$ then $G \underset{QI}{\simeq} G'$

Gromov's Program: QI classification of groups.

Coxeter groups in geometric group theory

Here is a list of examples where Coxeter groups have been used in connection to geometry/topology. See the introduction of Davis' book for a summary.

1. Creating exotic manifolds:
    - construct the first example of a closed aspherical (all homotopy groups trivial) manifold not covered by any $\mathbb{R}^n$
    - construct example of a closed aspherical manifold that is not residually finite
    - Construct example of a closed aspherical manifold that has unsolvable word problem
    - a closed aspherical topological manifold that is not homotopy equivalent to a smooth manifold
    - construct examples of Poincare duality groups that do not occur as π_1 of any closed aspherical manifold.
2. Hyperbolicity
    - construct examples of hyperbolic 4-manifolds (from Coxeter polytopes of reflection groups in $\mathbb{H}^4$)
3. Construction of CAT(0) spaces: a rich source of examples for groups acting on CAT(0) spaces.
4. Theory of special cube complexes
    - Davis complex of a Right angled Coxeter group is a cube complex. Any group that acts geometrically on a 'special cube complex' can be embedded into a RACG. This was a crucial step in the resolution of the virtual Haken conjecture.

Coxeter groups - definitions and preliminaries

Let S is a finite set. Let $M = (m_{st})$ be a symmetric $|S| \times |S|$ matrix with entries in $\{1, 2, \dots\} \cup \{\infty\}$ such that $m_{ss} = 1$ for all $s \in S$ and $m_{st} \geq 2$ for all $s \neq t$. Such a matrix is called a <u>Coxeter matrix</u>.

A group is called a <u>Coxeter group</u> if it admits a presentation of the form

$$< S \,|\, (st)^{m_{st}} \text{ for all } s, t \in S >$$

where (m_{st}) is a Coxeter matrix. Such a presentation is called a <u>Coxeter presentation</u>.

A <u>Coxeter system</u> (W, S) is a Coxeter group W such that there is a Coxeter matrix M defined with respect to S which gives a Coxeter presentation for W.

Example 0.1. There is only one possible Coxeter matrix on a singleton $S = \{s\}$ with corresponding group $\mathbb{Z}/2\mathbb{Z}$. Coxeter matrix on $S = \{s, t\}$ has a form $\begin{bmatrix} 1 & m \\ m & 1 \end{bmatrix}$ for $m \geq 2$. Consider the cases $m = 2, m = \infty$ and $m \neq \{2, \infty\}$. ∎

Recall the dihedral group $\mathcal{D}_n$ is the group of symmetries of a regular n-gon P_n in $\mathbb{E}^2$. Let r denote the rotation by $2\pi/n$ that generates the cyclic group of rotational symmetries $\mathcal{C}_n$ and let s be a reflection symmetry. Then

$$\mathcal{D}_n = < r, s \,|\, r^n, s^2, (sr)^2 > \cong \mathcal{C}_n \rtimes \mathcal{C}_2$$

Since $r = s\,sr$, letting $sr = t$ we have

$$\mathcal{D}_n = < t, s \,|\, (st)^n, s^2, t^2 >$$

There are two commonly used ways to represent Coxeter groups via graphs. Let M be a Coxeter matrix on S.

- <u>Coxeter graph/diagram</u> Γ_S: Vertex set is S and two vertices are joined by an edge if $m_{st} > 2$. Edges are labeled by m_{st} if $m_{st} \neq 3$. (Notice that no edge means $m_{st} = 2$.)
- <u>Presentation graph</u> Υ_S: Vertex set is S and two vertices are joined by an edge if $m_{st} < \infty$. Edges are labeled by m_{st} if $m_{st} \neq 2$. (Notice that no edge means $m_{st} = \infty$.)

Exercise 0.2.
a) Draw both the Coxeter diagram and the presentation graph for the following Coxeter matrices :

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & m \\ m & 1 \end{bmatrix} \text{ where } 3 < m < \infty, \begin{bmatrix} 1 & \infty \\ \infty & 1 \end{bmatrix},$$

b) How does a Coxeter group decompose if it is represented by a disconnected Coxeter diagram? a disconnected presentation diagram? A Coxeter group is called <u>irreducible</u> if it's Coxeter diagram is connected.

Exercise 0.3. A Coxeter group W is called rigid if there are two Coxeter systems (W, S) and (W', S') such that $W \cong W'$, then there is a label preserving graph isomorphism between Γ_S and $\Gamma_{S'}$.

a) Find two non-isomorphic Coxeter systems (equivalently, Coxeter diagrams) that determine the dihedral group D_6.

b) Determine which dihedral groups are rigid.

c) Prove that if Γ_S and $\Gamma_{S'}$ are two Coxeter diagrams that determine isomorphic groups and all edges of Γ_S are labelled ∞, then the same is true for all edges of $\Gamma_{S'}$.

Let (W, S) be a Coxeter system. Let $T \subset S$ be a finite subset. The subgroup W_T of W generated by T is called a special subgroup. Let Γ_T be the full subgraph of the Coxeter diagram Γ_S. It is not obvious but is true that (W_T, T) is itself a Coxeter system with Coxeter diagram Γ_T. Finite special subgroups of (W, S) are called spherical subgroups.

Exercise 0.4. A Coxeter group given by (W, S) is called a right-angled Coxeter group (RACG) if $m_{st} \in \{2, \infty\}$ for all $s \neq t$ in S.

a) Describe the graph property (in terms of both Coxeter graph and presentation graph) that characterizes those special subgroups of RACGs that are spherical.

Geometric reflection groups (examples of Coxeter groups)

A Euclidean convex polytope is the compact intersection of finitely many halfspaces in $\mathbb{E}^n$.

A spherical hyperplane is the intersection of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ (as the unit sphere with center the origin) with a linear hyperplane of $\mathbb{R}^{n+1}$. A spherical halfspace is the closure of a complement of a spherical hyperplane. A spherical convex polytope is the the compact intersection of finitely many spherical halfspaces in $\mathbb{S}^n$. A convex polytope is called an n-simplex if it has the combinatorial type of a simplex in $\mathbb{R}^{n+1}$.

A geometric reflection group is a group generated by reflections in the sides of a convex polytope P in $\mathbb{S}^n, \mathbb{E}^n$ or $\mathbb{H}^n$ such that P is a fundamental domain for the action. Such a polytope is called a Coxeter polytope. In these notes, we will primarily work with spherical and Euclidean polytopes. Please refer to Cashen's notes for the study of hyperbolic polytopes.

Lower dimension

Exercise 0.5 (One dimensional geometric reflection groups in $\mathbb{E}^1$)**.** Consider the real line with the Euclidean metric $\mathbb{E}^1$. The only non-singleton convex polytopes are intervals. Let P be one such interval, say $[-1/2, 1/2]$. Let $\rho_1 : x \mapsto -x-1$ and $\rho_2 : x \mapsto -x+1$ be the reflections about the faces $\{-1/2\}$ and $\{1/2\}$ respectively.

a) Show that the group generated by ρ_1 and ρ_2 is isomorphic to the infinite dihedral group $\mathcal{D}_\infty$.

b) Show that all geometric reflection groups defined on $\mathbb{E}^1$ are conjugate in the group of affine transformations of $\mathbb{E}^1$.

Exercise 0.6 (One dimensional geometric reflection groups in $\mathbb{S}^1$)**.** Show that geometric reflection groups in $\mathbb{S}^1$ are exactly the finite dihedral groups. (Hint: Construct an action of $\mathcal{D}_n$ on $\mathbb{S}^1$ and show that it is faithful.)

Theorem 0.7. *One-dimensional geometric reflection groups are in bijection with dihedral groups.*

Exercise 0.8 (Two dimensional geometric reflection groups in $\mathbb{S}^2$ and $\mathbb{E}^2$)**.** A Coxeter polygon is a Coxeter polytope in $\mathbb{S}^2, \mathbb{E}^2$ or $\mathbb{H}^2$ that is a fundamental domain for a geometric reflection group.

a) Show that the dihedral angle at each vertex of a Coxeter polygon is an even integral fraction of 2π.

Let $\mathbb{X}^2$ be $\mathbb{S}^2$ (resp. $\mathbb{E}^2$) with constant curvature = +1 (resp. 0). If P is a polygon in $\mathbb{X}^2$ with area $A(P)$, dihedral angles $\theta_i = \pi/m_i$ and Euler characteristic $\chi(P) = 1$, then the Gauss-Bonnet theorem says that

$$A(P)\kappa(\mathbb{X}^2) + \sum_{i=1}^{n} (\pi - \pi/m_i) = 2\pi\chi(P) = 2\pi.$$

b) Use the above formula to enumerate all possible Euclidean Coxeter polygons with the possible dihedral angles.

c) Use the above formula to enumerate all possible spherical Coxeter polygons with the possible dihedral angles.

d) *What can you say about hyperbolic Coxeter triangles using the above formula?

Exercise 0.9. A triangle group $\Delta(p, q, r)$ is a Coxeter group defined by the Coxeter matrix

$$\begin{pmatrix} 1 & p & r \\ p & 1 & q \\ r & q & 1 \end{pmatrix}$$

a) For $p, q, r < \infty$, when is $\Delta(p,q,r)$ a spherical/Euclidean/hyperbolic geometric reflection group?

b) Consider the action of $\Delta(2,2,\infty)$ on $\mathbb{E}^2$. What is the fundamental domain?

c) **Consider $\Delta(\infty,\infty,\infty)$ acting on $\mathbb{H}^2$. Draw the fundamental domain. Show that Δ has a finite index subgroup that is isomorphic to a free group. Show that Δ is not a geometric reflection group. Hint: Use the notion of quasi-isometry and quasi-isometry invariants.

Remark: Every geometric reflection group is a Coxeter group. There exist Coxeter groups that are not geometric reflection groups.

Theorem 0.10. *The two dimensional geometric reflections groups are as follows:*

- *spherical: spherical triangle groups*
- *Euclidean: Euclidean triangle group and* $\mathcal{D}_\infty \times \mathcal{D}_\infty$
- *hyperbolic:* $\cdots$

Higher dimension

Exercise 0.11. Show that the dihedral angles of a **Coxeter polytope** (those coming from a geometric reflection group) are integral submultiples of π.

Proposition 0.12. *1. Any spherical convex polytope with dihedral angles* $\leq \pi/2$ *(hence any Coxeter polytope) in* $\mathbb{S}^n$ *is a simplex.*

2. Any Euclidean convex polytope with dihedral angles $\leq \pi/2$ *(hence any Coxeter polytope) in* $\mathbb{R}^n$ *is a product of simplices. (In this case the geometric reflection group is a direct product and is called reducible.)*

Theorem 0.13. *A convex polytope* P *in* $\mathbb{S}^n$ *or* $\mathbb{R}^n$ *is a Coxeter polytope if and only if the dihedral angles of* P *are integral submultiples of* π *and if* P *is has the combinatorial type of a simplex in* $\mathbb{R}^{n+1}$*.*

Moreover, for dihedral angles $\theta_{ij} = \pi/m_{ij}$*, the Coxeter group determined by the Coxeter matrix* (m_{ij}) *is isomorphic to the group generated by reflections in the codimension 1 faces of* P*.*

Theorem 0.14. *All geometric reflection groups are Coxeter groups.*

Coxeter was able to enumerate all Coxeter polytopes in $\mathbb{S}^n$ and $\mathbb{E}^n$ to give a classification of irreducible geometric reflection groups in these setting. Recall, how we did this in the 2-dimensional case.

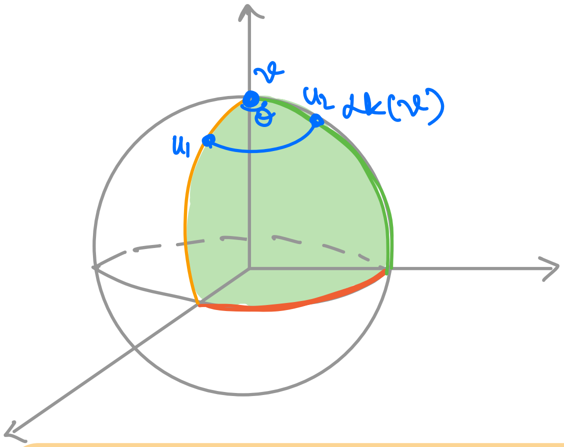
Next, we will focus on the spherical Coxeter polytopes, which in fact are always simplices.

Spherical polytopes

We now focus on spherical polytopes because they show up as "links" of vertices in convex polytopes in $\mathbb{S}^n/\mathbb{E}^n/\mathbb{H}^n$.
(Later: the links will dictate the geometry of a polyhedral complex).

The **link of a vertex** v of a convex polytope P in $\mathbb{X}^n$ ($\mathbb{S}^n/\mathbb{R}^n/\mathbb{H}^n$) is a convex polytope in $\mathbb{S}^{n-1}$ obtained by taking all unit vectors $\vec{u}$ in $T_v\mathbb{X}^n$ such that $\vec{u}$ is inward pointing

geod in $\mathbb{X}^n$ starting at v and initial vector $\vec{u}$ has non-trivial initial segment contained in P.



$L = lk(v,P)$ comes with a metric:

Example: $d_L(\vec{u}_1,\vec{u}_2)$ = distance in unit sphere in $T_v\mathbb{X}^n$ b/w $\vec{u}_1, \vec{u}_2$

= angle b/w $\vec{u}_1, \vec{u}_2$ = θ

= arc length b/w $\vec{u}_1, \vec{u}_2$

Prop A If P is a convex polytope in $\mathbb{S}^n$ with all dihedral angles $\leq \pi/2$, then P is a simplex.

GOAL: How to determine if a simplex is spherical given it's $\boxed{\text{dihedral angles.}}$?

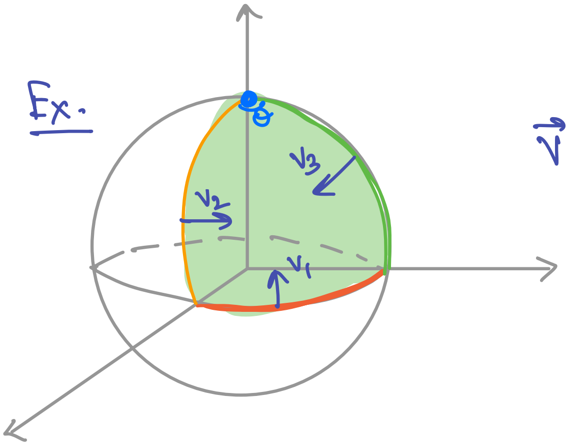
(i) Linear Algebra

Gram matrix of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is the matrix of their innerproducts

Gram matrix of a simplex := Gram matrix of the set of inward pointing unit normal vectors.

↳ the vector ⊥ to a halfspace.

Ex.



$\vec{v}_i = \vec{e}_i$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If θ_{ij} = dihedral angle b/w codimension 1 faces of σ ⊥ to $\vec{v}_i$ and $\vec{v}_j$.

then $\angle \vec{v}_i, \vec{v}_j = \pi - \theta_{ij}$

$$G_\sigma = \left(\cos \angle \vec{v}_i, \vec{v}_j \right) = \left(\cos(\pi - \theta_{ij}) \right) = \left(-\cos\theta_{ij} \right)$$

A real symm matrix M is **positive semi definite** if for all $v \in \mathbb{R}^n$ $\vec{v}^T M v \geq 0$.

positive definite if in addition $\vec{v}^T M \vec{v} = \vec{0} \Leftrightarrow \vec{v} = \vec{0}$

Sylvester's Criterion: $M_{n \times n}$ is positive definite

$\Longleftrightarrow$ for all $1 \leq i \leq n$, the upper left $i \times i$ submatrix M_i of M has positive determinant.

Prop B : Suppose $\theta_{ii}=\pi$, $\theta_{ij}=\theta_{ji}\in(0,\pi)$ for $i\neq j$
let $C=(-\cos(\theta_{ij}))$
There exists a spherical simplex with dihedral angles θ_{ij} iff C is positive definite.

Proof: $\Longrightarrow$

F1 σ is a spherical n-simplex $\Longleftrightarrow$ the set of inward pointing normal unit vectors $\{u_1, u_2, \dots u_{n+1}\}$ is a basis for $\mathbb{R}^{n+1}$.

F1 $\Rightarrow$ so there is $M\in GL(n+1,\mathbb{R})$
st. $Me_i = u_i$ (e_i std basis).

we have $\underset{u_i^T u_j}{\langle u_i,u_j\rangle} = \langle Me_i, Me_j\rangle = e_i M^T M e_j$

$\Rightarrow$ $C = M^T M$. Gram matrix.

to show: $w^T C w \geq 0$ with equality only for $w=\vec{0}$

$$\begin{aligned} &\overset{\shortparallel}{w^T M^T M w} \\ &\overset{\shortparallel}{(Mw)^T} Mw = \|Mw\| \geq 0 \end{aligned}$$

and $0 \Leftrightarrow w=0$ since $M\in GL_{n+1}$.

$\Longleftarrow$ C is positive definite

$\therefore$ all it's evalues are ≥ 0

and $eV=\vec{0}$ $\Leftrightarrow$ evalue $=0$.

also columns are linearly independent.

C real symmetric, so diagonalizing

$$C = S^T D S \quad \text{where } D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \ \lambda_n \end{bmatrix}$$

$\sqrt{C} = S^T \sqrt{D}\, S$ is well-defined.
and also positive definite.

∴ column vectors of $\sqrt{C}$ are linearly independent.
$\{v_1, \dots v_k\}$
∴ basis for some $\mathbb{R}^k \subset \mathbb{R}^n$.

F1 $\Rightarrow$ $\{v_1, \dots v_k\}$ are inward normal unit vector of some spherical simplex. σ

Also the Gram matrix of $\{v_1, \dots v_k\}$ is C.

F2 $\Rightarrow$ σ determined by C. ∎

F2. A spherical simplex is determined by its Gram matrix, equivalently, it's dihedral angles.

Proof of F1. $\Leftarrow$ suppose $\{u_1, \dots u_{n+1}\}$ is a basis of $\mathbb{R}^{n+1}$ of unit vectors.

$\exists\, \phi \in GL(\mathbb{R}^{n+1})$ st $\phi(e_i) = u_i$. $\quad \phi = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$

let σ_0 be the n-simplex given by $\{e_1, \dots e_{n+1}\}$.

then $\phi(\sigma_0)$ is also an n-simplex.

$\Longrightarrow$ σ n-simplex in S^n.
$\{u_1, u_2, \dots u_{n+1}\}$ has to be a linearly indep set of unit vectors.
$\Rightarrow$ basis.

Proof of F2

let σ_v be the simplex determined by inward pointing normal vectors $\{v_1, \dots, v_k\}$

let σ_u ——— u ——— $\{u_1, \dots, u_k\}$

s.t. $(\langle u_i, u_j \rangle) = (\langle v_i, v_j \rangle)$.

By **F1**. $\{v_1, \dots\}$ and $\{u_1, u_2, \dots\}$ are basis of $\mathbb{R}^{k+1}$

$\Rightarrow \exists\ \phi \in GL(\mathbb{R}^{k+1})$ s.t. $\phi(v_i) = u_i \ \forall i$.

Claim: ϕ is an isometry.

we have

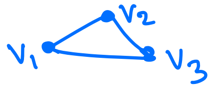
$$\begin{array}{c} \langle u_i, u_j \rangle = \langle \phi(v_i), \phi(v_j) \rangle \\ \shortparallel \\ \langle v_i, v_j \rangle \end{array}$$

$\Rightarrow \phi$ is an isometry. ($\{v_i\}$ basis).

ⓘ linear algebra: Polar dual

GOAL: How to determine if a simplex is spherical ? given its side lengths

Prop C. ∃ spherical simplex with edge lengths $l_{ij} \in (0, \pi)$



l_{ij} = length of edge joining v_i and v_j

$\Longleftrightarrow$

$C = (\cos l_{ij})$ is positive definite.

Definition: Let P be a convex spherical polytope of dim n with inward pointing unit normal vectors $U = \{\vec{u}_1, \dots u\}$ and vertices $\{\vec{v}_1, \dots\}$

It's Polar dual P^* is the convex spherical polytope (Prop.) of dimension n with inward normals V and vertices U

Prop D A convex spherical polytope P is a simplex $\Longleftrightarrow$ P^* is a simplex

Proof of Prop C $\Leftarrow$ Suppose C is +ve definite

Then $\sqrt{C}$ is also.

let V = set of column vectors of $\sqrt{C}$

Then the Gram matrix of $V = C$

C is +ve definite $\overset{\text{Prop B}}{\Longleftrightarrow}$ V is the set of inward pointing normals of a spherical simplex P.

$\underset{\text{Prop D}}{\Longleftrightarrow}$ iff V is the set of vertices of P^* (spherical simplex).

Now edge lengths of P^* are exactly l_{ij}. ∎

$\Rightarrow$ let P^* be the spherical simplex with vertices $V = \{v_1, \ldots v_k\}$ and edge lengths l_{ij}.

then $P^{**} := P$ has inward pointing normal vectors V.

Since P is also a spherical simplex

by Prop B its Gram matrix is +ve definite.

$$C = (\langle v_i, v_j \rangle) = (\cos l_{ij}).$$ ∎

SP1

Exercise: Show that $\exists$ spherical triangle with sides a, b, c iff $a+b+c < 2\pi$.

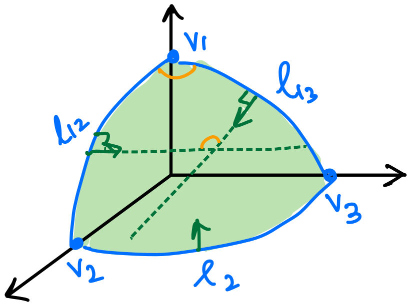
Corollary C

Prop E P convex spherical polytope
all edge length $\geq \pi/2$
Then P is a simplex

Proof: let $\{u_1, u_2, \dots u_n\}$ be the inward pointing normal vectors.
let $\{v_1, v_2, \dots \}$ be the vertices of P.

then $\angle v_i, v_j = l_{ij}$

let P^* be the polar dual of P.



Then

$$\begin{tikzcd}
P \text{ is a simplex} \arrow[d, Leftrightarrow, "\text{Prop D}"] & \\
P^* \text{ is a simplex} \arrow[r, Leftrightarrow] & C' = (\cos\angle v_i, v_j) \text{ is +ve definite.}
\end{tikzcd}$$

(inward normals are $\{v_1, \dots\}$)

$$\left(\cos l_{ij}\right) \overset{\text{def}^n}{=} \left(\cos(\angle v_i, v_j)\right) \overset{\text{geom}}{=} \left(\cos\left(\pi - \text{dihedral angle of } P^*\right)\right)$$

$l_{ij} \geq \pi/2 \Rightarrow$ dihedral angles of P^* $\leq \frac{\pi}{2}$.

$$\Big\Downarrow \text{ Prop A.}$$

P^* is a simplex ∎

(ii) Spherical Coxeter groups.

Prop F let (W,S) be a coxeter system with $S=\{s_i \mid i\in I\}$.
let $C = (-\cos \pi/m_{ij})$ be it's cosine matrix.

C is positive definite $\iff$

$\exists$ spherical simplex $\sigma \subset S^n \subset \mathbb{R}^{n+1}$ for $|I| = n+1$ whose Gram matrix is C
and such that $W \cong$ spherical simpl. reflect^n group with fundamental domain σ.

Proof: $\Leftarrow$ W sph simpl reflect^n Group then it's cosine matrix = Gram matrix of σ and hence +ve definite.

$\Rightarrow$ C postive definite $\overset{\text{Prop B}}{\Rightarrow}$ $\exists$ spherical simplex σ whose Gram matrix is C.

By Thm 2.9 $W \cong$ Geom reflect^n group in codim 1 faces of σ.

Thm.

For dihedral angles $\theta_{ij} = \pi/m_{ij}$, the Coxeter group determined by the Coxeter matrix (m_{ij}) is isomorphic to the group generated by reflections in the codimension faces of σ.

CAT(κ) geometry

Model space : For $\kappa \in \mathbb{R}$, $n \in \mathbb{N}$

M^n_κ unique, complete, simply connected, n-dim'l Riemannian mfld of const sectional curvature equal to κ.

$M^n_{-1} = \mathbb{H}^n$, $M^n_0 = \mathbb{E}^n$, $M^n_1 = \mathbb{S}^n$.

for $\kappa > 0$ $M^n_\kappa =$ sphere of radius $1/\sqrt{\kappa}$

for $\kappa < 0$ $M^n_\kappa =$ rescale metric on $\mathbb{H}^n$ by $1/\sqrt{|\kappa|}$.

Diameter of model space :

$D_\kappa = \infty$ if $\kappa \leq 0$

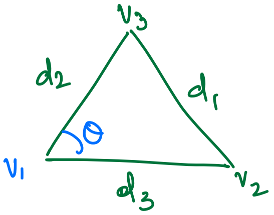
$D_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$

Prop: $\exists$ unique geod b/w two pts $x \neq y$ in M^n_κ when $d(x,y) < D_\kappa$.

C1.

Exercise: let X be a geod metric space
consider a geod triangle Δ.
then $\exists!$ geod triangle Δ' in M^2_κ with the same (called companion Δ) side lengths.

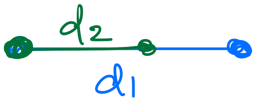
Proof:



$|d_j - d_k| \leq d_i \leq d_j + d_k$
for $i \neq j \neq k \in \{1,2,3\}$.

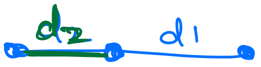
$k=0$.
-1.

extreme 1.

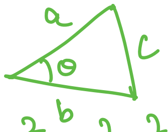


$x = d_1 - d_2$.

extreme 2.



$x = d_1 + d_2$.



$c^2 = a^2 + b^2 - 2ab\cos\theta$

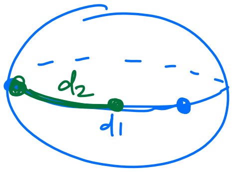
$\leftarrow$ [$x \in [d_1 - d_2, d_1 + d_2]$ varies continuously

d_3 satisfies this.

$< \text{ or } \leq$

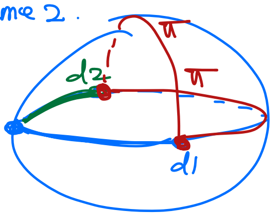
- $k = 1$. $d_i \leq \pi$. $d_1 + d_2 + d_3 \boxed{\leq} 2\pi$ $2D_\kappa$.

extreme!



then $x = d_1 - d_2$.

extreme 2.

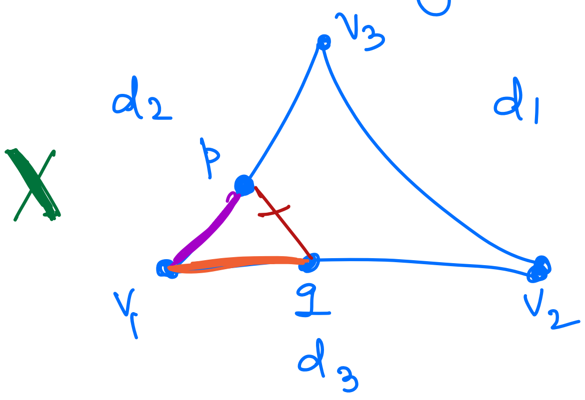


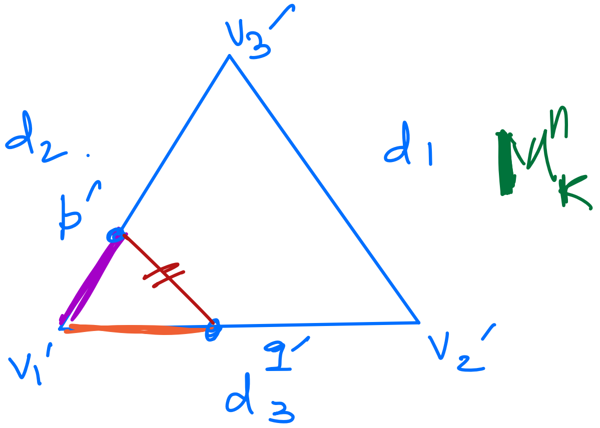
if $= 2\pi$ then

then $x = 2\pi - (d_1 + d_2)$.

x varies continuously b/w $[d_1 - d_2, 2\pi - (d_1 + d_2)]$
d_3 satisfies this condition!

A geod metric space X is called a CAT(k) space if
if for every geod Δ in X
and corresponding comparison Δ' in M^n_k
the following is true:





$$d_X(p,q) \le d_{M^n_k}(p',q').$$

slogan: triangles are at most as fat as in model space M^2_k.

Example: $\mathbb{S}^n$ is CAT(1)
$\mathbb{E}^n$ is CAT(0)
$\mathbb{H}^n$ is CAT(−1).

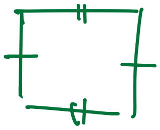
CAT stands for Cartan, Alexandrov, Toponogov.
Gromov coined this term.

"A complete Riemannian mfld has curvature ≤ k in the above sense if and only if all of its sectional curvatures ≤ k."

A space has curvature at most k, or is locally CAT(k) if every point has a CAT(k) neighborhood.

Theorem A complete geod metric space of curvature at most k, is CAT(k) $\Leftrightarrow$ it has no isometrically embedded loops of length $< 2D_k$.

Ex. Torus $\mathbb{T}^2$ locally CAT(0) but not CAT(0).



it's universal cover is CAT(0).

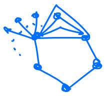
Cartan-Hadamard theorem:

X a complete connected geodesic metric space
X locally CAT(k) for $k \leq 0$
then $\tilde{X}$ (universal cover) is a CAT(k) space.

C2 Exercise: show that a tree is CAT(κ) for any $\kappa \in \mathbb{R}$.

C3 Exercise: If X is CAT(κ) for $\kappa < \kappa'$ then X is also CAT(κ') (But not vice versa).

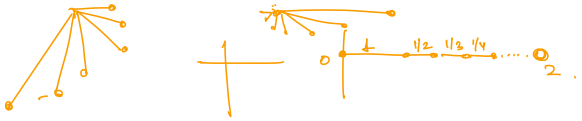
Exercise: C4 Show that a graph with only finitely different edge lengths is CAT(1) $\iff$ it does not contain an isometrically embedded cycle of length $< 2\pi$

Pf. Graph 

locally CAT(1)? nbd of a pt. is a star tree  which is CAT(1).

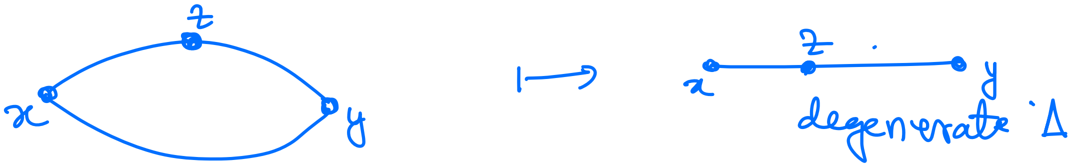
$\therefore$ By thm ✓

complete geod metric space
every Cauchy seq has a cvg subseq.



C5

Exercise: Show that there is a unique geod between two points at dist $< D_k$ in a CAT(k) space.
(CAT(0) spaces are uniquely geodesic).



degenerate Δ

C6

Exercise: let X be a CAT(0) space.
Show that local geodesics are geodesics.

$\gamma: [0,\infty) \to X$ a path is called a local geod
if $\forall t \geq 0$, $\exists \varepsilon$ st $\gamma|_{(t-\varepsilon, t+\varepsilon)}$ is a geodesic.

Hint: $G = \{ t \geq 0 \mid \text{st } \gamma|_{[0,t]} \text{ is a geod} \}$.
$G \neq \phi$ b/c γ is a geod in a nbd of 0.
show G closed
show G open

C7

Exercise: A map $\phi : X \to Y$ b/w two metric spaces is called a local isometry if $\forall x \in X$, $\exists$ nhd B_x of x s.t. $\phi|_{B_x}$ is an isometric embedding.

If $\phi : X \to Y$ is a local isometry from
X geod metric space to
Y a CAT(0) space
then show that ϕ is an isometric embedding.

Thm If G acts geometrically on a complete CAT(0) space, then G has only finitely many classes of finite subgroups.

proof sketch:

f1. Every bounded set in a CAT(0) space has a unique \`center.'

cocompact act^n $\Rightarrow$ compact set K that is a fundamental domain

properly dist $\Rightarrow$ $\#\{g \in G \mid gK \cap K \neq \phi\}$ is finite.

$\therefore$ $\exists$ finite set $F \subset G$ of elements that fix a point of K.

Now if H is a finite subgroup of G then $H \cdot x$ is bounded for any $x \in X$.

$\therefore$ $\exists$ center x_0 that is fixed by H.

now $gx_0 \in K$ for some $g \in G$

$\therefore$ $gH\bar{g}$ fixes gx_0 and is contained in the finite set F

$\therefore$ every finite subgroup of G is conjugate into F.

Consequences of acting on a CAT(0) space.

$G \curvearrowright X$ CAT(0)

`geometric action'

Algebraic :

- G is finitely presented
- G has finitely many conjugacy classes of finite subgroups

has a f.i subgroup

- solvable subgroups are virtually abelian
- Every abelian subgroup is f.g.

Algorithmic : There are algorithms to solve

- word problem
- conjugacy problem

Geometric :

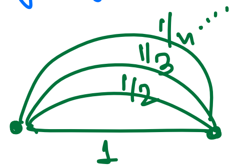
- exponential volume growth
- Dehn function is atmost quadratic.

M_k- polyhedral complex

For $k \in \mathbb{R}$, an M_k- polyhedral complex is a space X built from a disjoint union of convex polyhedra $P_i \in M_k^{n_i}$ by identifying some polytopes isometrically along faces.

Fact: An M_k- polyhedral cmplx built from finitely many isometry type of polytopes is a complete geod metric space.

Ex.



Not even a metric space!

An M_k-polyhedral cmplx is built out of CAT(k) pcs

when is it itself CAT(k)??

nhd of a point in the interior of a polytope is CAT(k)

at vertices?

An M_k-poly cmplx satisfies the link condition if the lk of every vertex is a CAT(1) space.

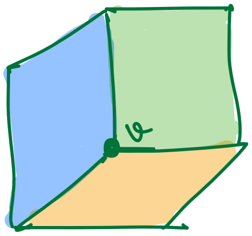
Thm. An M_k-poly complex with finitely many isometry types of polytopes

Thm ① has curvature at most k $\Longleftrightarrow$ it satisfies the lk condition

Thm ② · is CAT(k) $\Longleftrightarrow$ ⓐ satisfies the link condition

"girth condit" ⓑ does not contain isometrically embedded loop of length $< 2D_k$.

Example: (i)

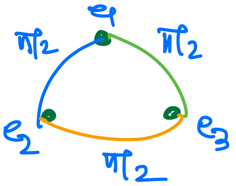


M_0-polyhedral complex.

- Fails the lk condition

∴ not CAT(0) space.

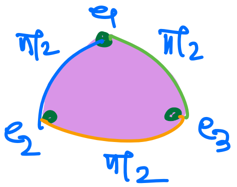
$lk\, v =$



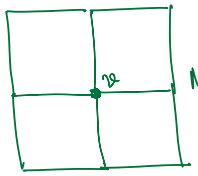
Is this a CAT(1) space?

No!

comparison $\Delta' =$

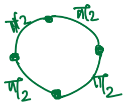


(ii) $X =$



M_0-poly comp.

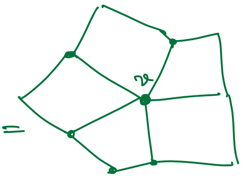
$lk =$



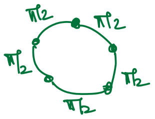
is CAT(1).

X is CAT(0) ✓

(iii) $X =$



$lk =$



is CAT(1).

X is CAT(0) ✓

Davis Complex construction

Recall $\{\text{geometric reflection groups}\} \subsetneq \{\text{Coxeter grps}\}$

let W be a geom. ref grp

then $W \underset{\substack{\text{prop disc.}\\ \text{cocompactly}}}{\curvearrowright} \mathbb{S}^n / \mathbb{E}^n / \mathbb{H}^n$.

let W be a Coxeter group that is NOT a geom ref grp.

GOAL: Find a 'nice' space Σ_W (called Davis complex).

st $W \underset{\substack{\text{prop. disc.}\\ \text{cocompactly}}}{\curvearrowright} \Sigma_W$.

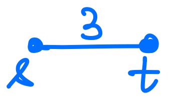
CONSTRUCTION: Given (W, S).

let
- $\mathcal{S} = \{ T \subseteq S \mid W_T \text{ is spherical} \}$
- $\mathcal{S}_{>\phi} = \mathcal{S} \setminus \phi$
- $W\mathcal{S} = \bigcup_{T \text{ spherical}} W/W_T$.

Each of the above sets are partially ordered by inclusion.

let the corresponding simplicial cmplxes be:
- K the chamber
- L the nerve or link
- Σ the Davis complex.

Example: $(W,S) =$

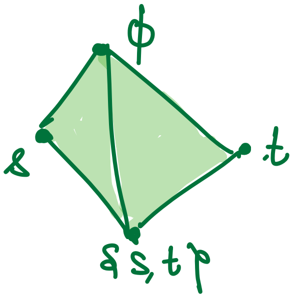


$\langle s,t \mid s^2, t^2, (st)^3 \rangle$.

$\mathcal{S} = \{\phi, \{s\}, \{t\}, \{s,t\}\}$

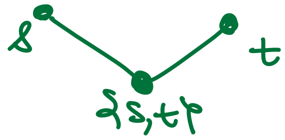
$\mathcal{S}_{>\phi} = \{\{s\}, \{t\}, \{s,t\}\}$.

$K =$

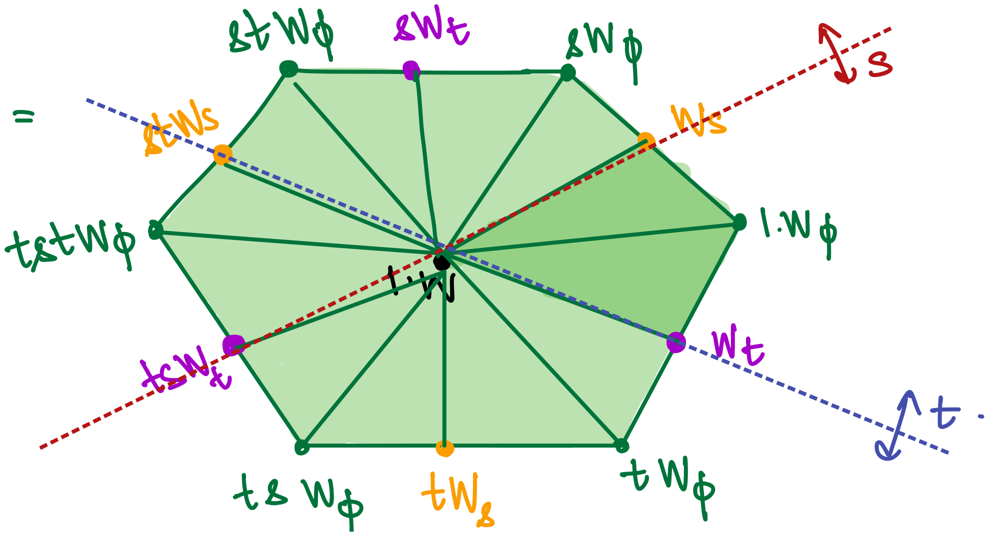


- ● $W/W_\phi = \{1, s, t, st, sts\}$.
- ● $W/W_s = \{s, ts, sts\}$
- ● $W/W_t = \{t, st, tst\}$
- ● $W/W_{\{s,t\}} = \{1\}$.

$L =$



$\Sigma_W =$



Observation:

1. Action of $W = D_3$ on Σ_W.
   s, t act by reflections
2. Fundamental domain for the action is K.
3. lk of W_ϕ is L.

- Too many triangles!

RECELLULATION :

For every $T \subset S$ st. W_T is spherical
and for every $g \in W$

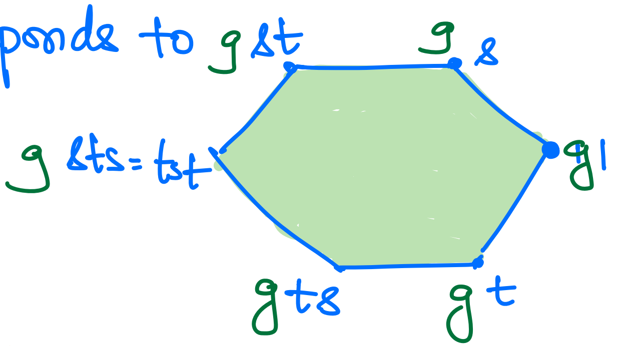
$$\Sigma'_W = \text{built by gluing cells of dim } |T| \text{ for each } gW_T.$$

Vertices : $T = \phi$, $W_T = 1$,
correspond to gW_ϕ or elements of W.

Edges : $T = \{s\}$, $W_T = \{1, s\}$.
correspond to $g \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, gs$ for $gW_{\{s\}}$.

- 1-skeleton of Σ'_W is exactly $Cay(W,S)$.

2-cells : $T = \{s,t\}$, $W_T = \langle s,t \mid s^2, t^2, (st)^m \rangle$ $m < \infty$.
corresponds to

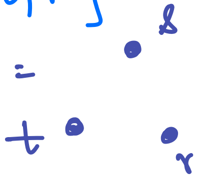


for gW_T.

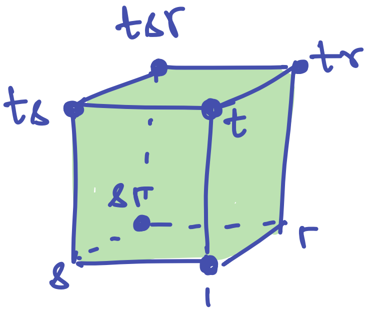
- 2-skeleton of Σ'_W is exactly Cayley complex for (W,S).

3-cells: $T = \{s,t,r\}$

Ex. $\Gamma =$



$W_T = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$



Question: How to glue the cells together?

We will specify a metric on these cells and then glue by isometry on faces.

METRIZATION

To show: Each cell can be given a Euclidean metric.

n-cell $\equiv$ W_T spherical, for $|T| = n$

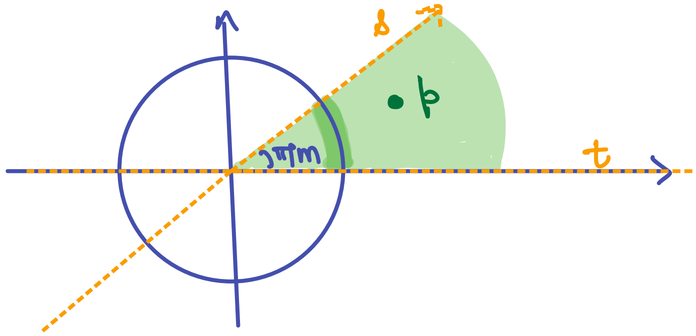
$\Updownarrow$

finite geom reflection groups $\curvearrowright S^{n-1}$
with fund. domain a simplex.

$\Downarrow$

upto conjugation by element of $\mathrm{Isom}(\mathbb{R}^n)$
wma $W_T \subset O(n)$
and $W_T \curvearrowright \mathbb{R}^n$ with fundamental domain a simplicial cone C_T.

Ex. $W_T = \underset{s}{\bullet}\overset{m}{\text{———}}\underset{t}{\bullet}$



Now for any $p \in C_T$, $W_T \cdot p$ is a Euclidean polytope whose combinatorial type is indep of p.

p is specified by $\bar{d} \in (0,\infty)^T$
where $\bar{d} = \{ d_t \mid t \in T \}$
↖ dist b/w p and hyperplane for $t \in T$.

Gluing n-cells

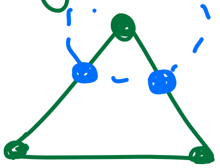
Fix $\bar{d} \in (0,\infty)^S$

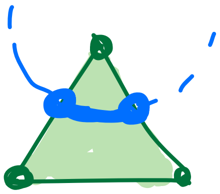
$\forall\, T \subseteq S$ get a Coxeter cell $\Sigma_T(\bar{d}_T)$ for each W_T.
and glue $g \cdot \Sigma_T(\bar{d}_T)$ by isometry along faces

Theorem: (Davis / Moussong)
for any Coxeter system (W,S) and choice $\bar{d} \in (0,\infty)^S$ the Davis complex $\Sigma = \Sigma(W,S)$ admits the str of a Euclidean polyhedral complex st

- $V(\Sigma) \leftrightarrow W$
  $\Sigma^{(1)} \cong \mathrm{Cay}(W,S)$
  $\Sigma^{(2)} \cong$ Cay cmplx for Coxeter presentat^n (W,S)
- each n-cell corresponds to gW_T where T spherical of size n and the cell is $\Sigma_T(\bar{d}_T)$
- lk of every vertex is $L(W,S)$
- Poset of cells of $\Sigma \cong W\mathcal{S}$.

lk of a vertex $\equiv$ spherical nhd of that point.





Exercises:

D1 Show that K is a fundamental domain for the act^n of $W \curvearrowright \Sigma$

D2. Show that $W \curvearrowright \Sigma$ prop disc. and cocompactly

D3. show that Σ is simply connected.

D4. Suppose (W,S) is a Coxeter system such that every proper special subgroup is finite. Show that K is a simplex.

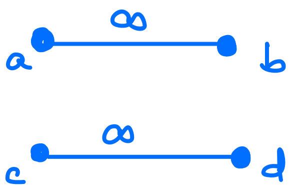
Exercise

D5.

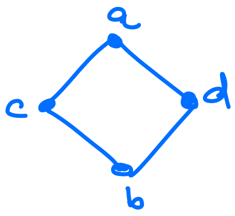
RACG.

$W = D_\infty \times D_\infty$

$\Gamma =$

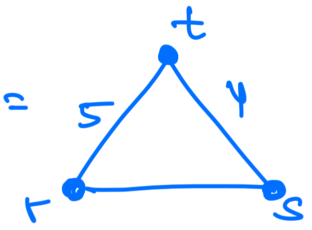


$\Gamma =$

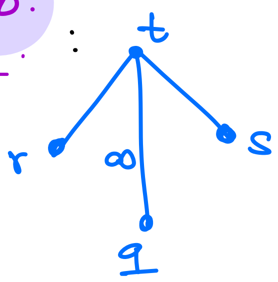


1. Draw K, L, Γ presentation graph.
2. Do you see a relationship between L and Γ.
2. Write $\mathcal{S} = \{T \subseteq S \mid T \text{ spherical}\}$.
3. Draw cells $\Sigma_T(d_T)$.
4. glue using L.

Exercise **D6.** $\Gamma =$

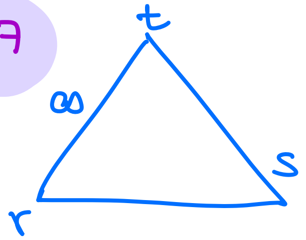


Exercise **D8.** :



D7

Exercise



Davis complex is CAT(0)

For (W,S) a Coxeter system, let Σ_W (or Σ) be the Davis complx

we already saw,

① Σ is simply connected

② $W \curvearrowright \Sigma$ prop disc and cocmptq.

FLOW CHART.

Σ is CAT(0) $\Longleftrightarrow$ ① Σ is s.c.

② Σ satisfies the link condition $L(W,S)$ is CAT(1)

$\Updownarrow$ Moussong's Lemma.

(2a) L has edges of length $\geq \pi/2$

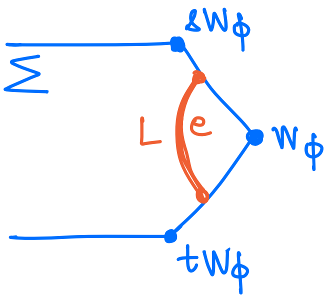
(2b) L is a metric flag complex

↳ If some set of edges of L can be the 1-skeleton of a spherical simplex then $\exists$ spherical simplex in L whose 1-skeleton is those edges.

(2a) To show: L has edges of length $\geq \pi/2$

Recall L = simplicial complex corresponding to the poset $\{T \neq \phi,\ T \subset S \text{ spherical}\}$.

|| vertices of L correspond to $\Sigma_{\langle s \rangle}$ for $s \in S$.
|| Edges of L correspond to $\Sigma_{\langle s,t \rangle}$ for $s,t \in S$.



length of e = dihedral angle b/w $\Sigma_{\langle s \rangle}$ and $\Sigma_{\langle t \rangle}$ for $s,t \in S$.

$$= \pi - \frac{\pi}{m_{st}} \geq \pi/2.$$

ⓑ2b **To show:** L is a metric flag complex.

pf: in other words, to show that L is determined by it's one-skeleton.

Suppose $T \subset S$ s.t. the vertices $\Sigma_t \in L$ are pairwise joined by edges.

Let l_{st} = length of each edge $= \pi - \dfrac{\pi}{m_{st}} \geq \pi/2$.

Define $c_{st} = \cos l_{st} = -\cos(\pi/m_{st})$

We want to show that there is a spherical simplex with edge lengths l_{st}.

$\exists$ spherical simplex in L with edge lengths $l_{st} = \pi - \pi/m_{st}$

see next page ← | Prop C $\iff$ C is positive definite.

| Prop F $\iff$ (W_T, T) is a sph. geom ref group.

def^n $\iff$ $\Sigma_{\langle T \rangle}$ is a cell of Σ

def^n $\iff$ $\{\Sigma_t \mid t \in T\}$ spans a cell/simplex in L.

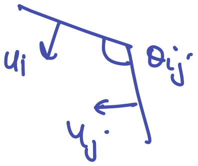
∎

Recall Prop: P convex polytope

- inward pointing normal vectors $\{\vec{u}_1, \vec{u}_2 \dots \vec{u}_n\}$

  Gram matrix $C_1 = \left(\langle \vec{u}_i, \vec{u}_j \rangle \right) = \left(\cos \angle \vec{u}_i, \vec{u}_j \right)$

- dihedral angles b/w codimension-1 faces θ_{ij}



  $\theta_{ij} = \pi - \angle u_i u_j \Rightarrow$

  $C_2 = \left(-\cos \theta_{ij} \right).$

  $[C_1 = C_2]$

- lengths of edges of P. l_{ij} st $l_{ii} = 0$

  $C_3 = \left(\cos l_{ij} \right)$

Prop B P is spherical simplex $\Longleftrightarrow$ $C_1 = C_2$ is +ve def

prop C P is a spherical simplex $\Longleftrightarrow$ C_3 is +ve def.

Cor C $a, b, c \in (0, \pi)$ are the sides of a spherical triangle $\Longleftrightarrow$ $a+b+c < 2\pi$

Recall Thm: Prop F

(W_T, T) Coxeter group with C = cosine matrix of (W_T, T)

$\theta_{ss} = \pi$

$\theta_{st} = \pi - \pi/m_{st}$

$C = \left(-\cos(\theta_{st}) \right).$

Then C is positive definite.

$\Longleftrightarrow$ $\exists$ spherical simplex σ whose Gram matrix is C.

st. W_T isom to the spherical geom reflect^n grp with fundamental dom. σ.

One direction of Moussong's lemma

GOAL: L M_1-polyhedral complex
L has all edges of length $\geq \pi/2$.
L is CAT(1)
Then L is a metric flag complex

Proof: By **Prop E**, we have that L is in fact a spherical simplicial complex.

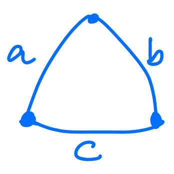
Suppose L is not metric flag.
let σ_n be a spherical simplex st. $\sigma_n^{(1)} \subset L$ but $\sigma_n \not\subset L$.
Also assume all faces of σ_n are contained in L, else pass to a lower dimensional simplex.
then for a vertex v of σ_n

$L_{n-1} := lk(v,L)\big|\partial\sigma_n$
- is a spherical simplicial cmplx.
- is CAT(1)
- is not metric flag.
- $\dim(L_{n-1}) \lneq \dim(L = L_n)$

$(*)$

$\exists\ \sigma_{n-1}^{(1)} \subset L_{n-1}$ the 1-skeleton of a spherical simplex without the simplex
then $\exists\ v \in \sigma_{n-1}^{(1)}$ st
$L_{n-2} := lk(v, L_{n-2})\big|\partial\sigma_{n-1}$ satisfies $(*)$

going down dimension, we end up at



$\sigma_0^{(1)}$: 1-skeleton of a spherical 2-simplex in a CAT(1) spherical simplicial cmplex of dimension 1.

Now σ_0 is a spherical 2-simplex $\overset{\text{Cor C}}{\iff}$ $a+b+c < 2\pi$
$\underset{\text{CAT(1)}}{\iff}$ $\sigma_0^{(1)}$ is filled in
in a 1-dim'l space.

- σ_n n-dim'l simplex.

  $lk(v, \sigma_n) \cong \sigma_{n-1}$ $(n-1)$ dim'l simplex.

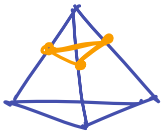
- $\sigma_n^{(n-1)}$ $(n-1)$-skeleton of a n-simplex (sph) $\subset L$

  $lk(v, \sigma_n^{(n-1)}) \cong \sigma_{n-1}^{(n-2)}$ $(n-2)$ skeleton of a $(n-1)$-simplex. (sph)

  $\downarrow$ keep going to get

  $n=3$.

  $\sigma_2^{(1)}$ 1 skeleton of a 2-simplx (sph).



let a,b,c be the sides of $\sigma_2^{(1)}$.

then $\sigma_2^{(1)}$ is the 1-skeleton of a spherical 2-simplx $\overset{\text{Gr G}}{\Longleftrightarrow}$ $a+b+c<2\pi$

$\Updownarrow$ CAT(1) for 1-dim.
each iterated lk is CAT(1)
$\therefore$ no short-loop.

$\sigma_2^{(1)}$ is filled in.

$\Updownarrow$

$\sigma_3^{(2)}$ is filled in

$\vdots$ $\Updownarrow$

$\sigma_n^{(n-1)}$ is filled in

Other direction of Moussong's Lemma.

GOAL: L M_1-polyhedral cmplx
all edges have length $\geq \pi/2$.
L metric flag complex
Then L is CAT(1).

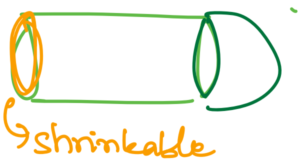
Recall / record some results

Thm G. An M_1-polyhedral complex is CAT(1)
$\iff$ "lk condition" link of every vertex is CAT(1)
+ "girth condit^n" no isometrically embedded closed circle of length $< 2\pi$.

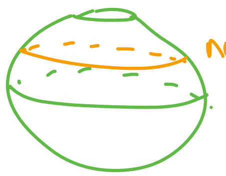
Prop H [Bowditch] X cmpt metric space, curvature at most 1, every closed rectifiable curve of length $\underset{\cancel{-}}{<} 2\pi$ is shrinkable, then X is CAT(1).

shrinkable: homotopic to a strictly shorter curve thru a homotopy of length non-increasing curves.

Ex.



Ex.



Prop I Properties inherited by links:

ⓐ spherical simplicial complex with length of edges $\geq \pi/2$

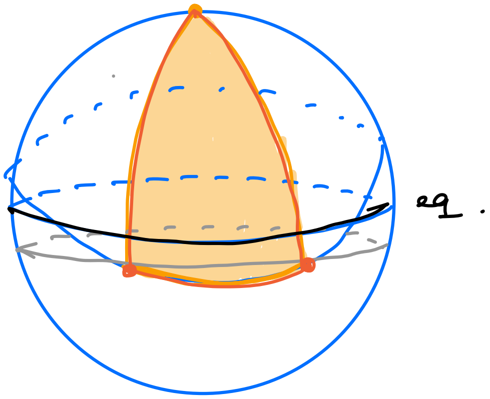
ⓑ metric flag complex.

Prop J σ spherical simplex with edges of length $\geq \pi/2$

v_0 a vertex of σ

e an edge opposite to σ

Then $d(v_0, e)$ is realised by some edge of σ.



Crux of the proof $l_e \geq \pi/2$ + metric flag + spherical $\Rightarrow$ CAT(1).

Spherical simplices start to "bulge" once their edge lengths are longer than $\pi/2$.

This forces geodesics to follow the one skeleton

But now a loop of length at most 2π cannot cross more than 3-edges, forming a Δ

Flag condition implies that this Δ is a simplex and hence there are no short geodesic loops.

Proof $l_e \geq \pi/2$ + metric flag + spherical $\Rightarrow$ CAT(1).

Suppose L is not CAT(1)

lk condition fails
some lk of L is not CAT(1) L_1

girth condition fails

lk condition fails
some lk of L_1 is not CAT(1)

girth condition fails

⋮

lk condit^n fails
L_n = lk of L_{n-1} is 1-dim'l. and not CAT(1)

$\equiv$ girth condition fails for L_n.

So wma, we have an iterated lk L'

st L' is a
- spherical cmplx
- all edges of length $\geq \pi/2$
- L' is metric flag.

} Prop I

- all lks of L' are CAT(1)
- L' fails the girth condition.

Failing girth condition means:

∃ isometrically embedded circle γ_0 of length $< 2\pi$

let L'' be the union of all closed cells that γ_0 intersects.

Then L'' is compact and locally contractible.

↓ b/c there are finitely many isometry type of cells. So for instance we don't have a shrinking wedge of circles.

By Bowditch Prop **H**, ∃ closed rectifiable curve γ of length $< 2\pi$ in L'' that is not shrinkable.

Since L'' is compact and locally contractible, wma γ is

- minimal length among all non-trivial, non-shrinkable loops
- locally geodesic.
  → if not, then ∃ local homotopy to a shorter curve. (since L'' locally contractible)

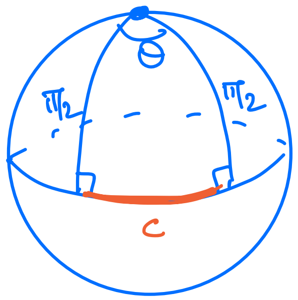
Rename L'' as L' for the rest of the argument.

let v be a vertex of L'. Recall $\frac{\pi}{2} \le l_e < \pi$

We have

① $\underbrace{\overline{B(v,\pi/2)}}_{}$ is isometric to $\underbrace{\text{spherical cone on } lk(v)}_{}$.

obs:

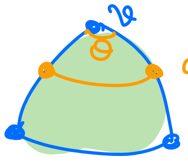


claim $c=\theta$

law of cosines:

$$\cos c = \cos a \cos b + \sin a \sin b \cos\theta$$
$$= \cos \pi/2 \cos \pi/2 + \sin \pi/2 \sin \pi/2 \cos\theta$$
$$\cos c = 0 + 1.\cos\theta$$
$$\Rightarrow c=\theta \quad (\text{since } \theta<\pi).$$

Recall metric on $lk(v)$



$lk(v) \subset S^1$

unit vectors that point inside the polytope.

metric coming from S^1.

② $\overline{B(v,\pi/2)} \subset \underbrace{\text{closed star of } v}_{\text{union of all closed cells that contain } v.}$

③ L' is covered by open balls of radius $\pi/2$ around it's vertices.

Pf: Since $\frac{\pi}{2} \le l_e < \pi$, edges are covered.

let σ_n be a simplex in L' that is not covered by these open balls around its vertices.

then $\exists\, p \in \sigma_n$ which is dist $\geq \pi/2$ from all vertices

isometrically embed σ_n in $\mathbb{R}^{n+1}/S^n$, st. p is the north pole.

then all the vertices are in the closed southern hemisphere.

$\Rightarrow \sigma_n \subset$ southern hemisphere $\rightarrow\leftarrow$. ∎

④ $\underbrace{\text{Nerve of the covering}}$ of L' by $B\underset{\text{open}}{(v, \pi/2)}$ is $L'^{(1)}$.

$N(\mathcal{C})$ = simplicial complex

v : opensets in the covering

e : intersectⁿ of 2 open sets

k-simplex : mutual intersectⁿ of k+1 open sets.

Pf. $L'^{(1)} \subseteq N(\mathcal{C})$ b/c $l_e < \pi$

Now suppose $\exists\, v, v' \in L'^{(0)}$ st $d_{L'}(v, v') < \pi$

and $B_v \overset{}{\bullet\!\!-\!\!-\!\!\bullet} B_{v'}$ in $N(\mathcal{C})$

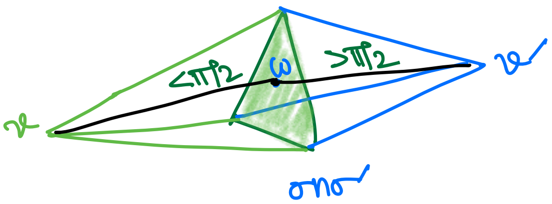
But v, v' are not in a common simplex

($\equiv$ no edge b/w them)

let $w \in B(v) \cap B(v') \subseteq \overline{st(v)} \cap \overline{st(v')}$

$\Rightarrow \exists$ simplex $\sigma \in \overline{st(v)}$ st. $\sigma \cap \sigma' \ni w$.

$\sigma' \in \overline{st(v')}$



By Prop J, $d(v, \sigma\cap\sigma')$ is realized by an edge

$d(v, w) \geq d(v, \sigma\cap\sigma') \geq \pi/2$

which is a contradictⁿ.

$\Rightarrow$ v, v' belong to the same simplex and are hence adjasent. ∎

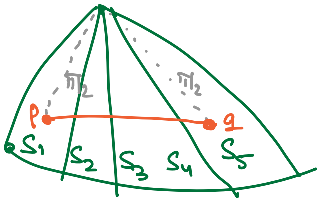
⑤ γ can be homotoped to be contained in $L^{'(1)}$.

(5a) More precisely, consider $\delta = \gamma \cap B(v, \pi/2)$

∃ length non-increasing homotopy of δ rel endpoints to a path that goes through v

in fact a local geodesic: if not a local geod, then we can homotope it to be shorter which is a contradict^n since we chose a minimal one.

proof: let S = union of 2-simplices containing v that intersect δ.

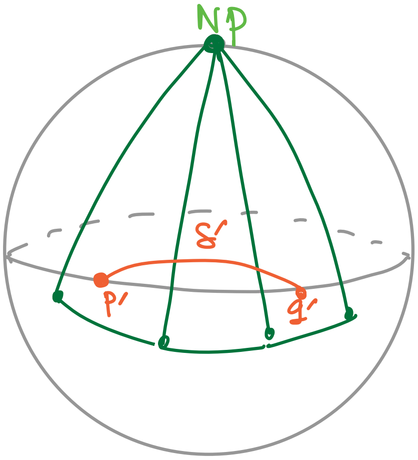


since $\ell(\delta) \leq \pi$, ∃ local $\overset{\phi}{\text{isometry}}$ from S into the northern hemisphere of S^2.

first map S_1 into S with v as the NP.

now map S_2 st map compatible on $S_1 \cap S_2$

.....



δ' is a local geodesic in the interior of northern hemisphere.

⇒ p', q' antipodal points

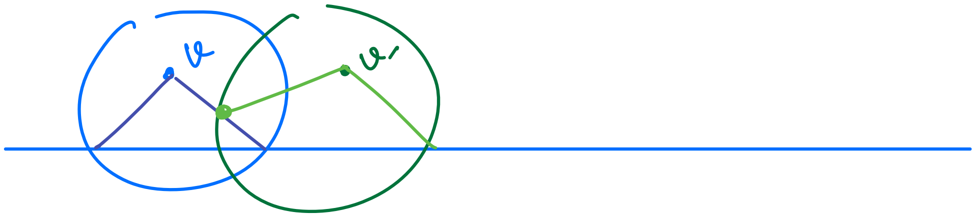
now take constant-length homotopy b/w δ' and δ'' ↘ geod b/w p', q' passing through NP

pull it back to get a local geod joining p, q ~~&~~ via v and in a length non-increasing manner. ∎

(5b) Apply (5a) to each open ball along the length of γ.

Claim: The new curve γ' is
- locally geod ✓
- non-shrinkable
- has length $< 2\pi$

Pf. length non-inc homotopy.

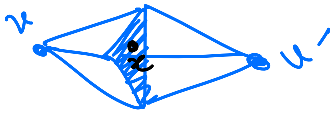


(5c) γ' is contained in the 1-skeleton $L'^{(1)}$.

let $x \in B(v) \cap B(v') \cap \gamma'$

then by construction $\exists$ local geod joining x and v

→ " —— " x and v'



as in (4) we conclude that

v, v' are in the same simplex

and the unique locally geod path joining

v, v' is the edge b/w them.

⑥ Final conclusion

γ' loop of length $< 2\pi$, non-shrinkable, $\gamma' \subset L'^{(1)}$

$l_e \geq \pi/2$

$\Rightarrow \gamma'$ consists of at most 3 edges

But L' metric flag complex $\Rightarrow$ a triangle of total length $< 2\pi$ is filled in.

$\rightarrow\leftarrow$ γ' is non-shrinkable ∎

∎

Exercise: CGD 1. Show that every finite subgroup of a coxeter group (W,S) is conjugate into a spherical subgroup.

Thm CGD 2. Every solvable subgroup of (W,S) is f.g. and virtually abelian.

Thm CGD 3. There are finitely many conjugacy classes of finite subgroups of (W,S).