Geometry of Coxeter Groups

Exercise sheet

mini-course at Frontiers in Mathematics
Kerala School of Mathematics
(KSOM)

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GGT

Ex. GGT 1 A metric space is QI to a point $\Leftrightarrow$ it has finite diameter.

Let $(\mathbb{Z}, d)$ be the induced metric on $\mathbb{Z}$ from $\mathbb{R}$
then $i : \mathbb{Z} \longrightarrow \mathbb{R}$ is a quasi-isometry.
(isometric emb, coarsely surjective).

Exercise: GGT 2. Every f.g. group is a metric space, well defined upto quasi-isometry.

GGT3 Let X be a CW complex, which is locally finite and finite dim'l.
Suppose a f.g. group G acts on X combinatorially.
(sends n-cells to n-cells)

Suppose cell stabilizers are finite.
Can you conclude that $G \curvearrowright X$ is properly discontinuous?

GGT4 **Ex.** $\mathbb{F}_2 \cong_{QI} \mathbb{F}_3$ $\quad \mathbb{F}_3 \curvearrowright T$ properly, cocompactly.

GGT5 **Ex.** $G \overset{\cdot}{\leq} G'$ then $G \underset{QI}{\simeq} G'$
finite index

SP1

Exercise: Show that $\exists$ spherical triangle with sides a, b, c iff $a+b+c < 2\pi$.

Coxeter groups

Exercise 0.2. a) Draw both the Coxeter diagram and the presentation graph for the following Coxeter matrices :

$$[1], \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & m \\ m & 1 \end{bmatrix} \text{ where } 3 < m < \infty, \begin{bmatrix} 1 & \infty \\ \infty & 1 \end{bmatrix},$$

b) How does a Coxeter group decompose if it is represented by a disconnected Coxeter diagram? a disconnected presentation diagram? A Coxeter group is called irreducible if it's Coxeter diagram is connected.

Exercise 0.3. A Coxeter group W is called rigid if there are two Coxeter systems (W, S) and (W', S') such that $W \cong W'$, then there is a label preserving graph isomorphism between Γ_S and $\Gamma_{S'}$.

a) Find two non-isomorphic Coxeter systems (equivalently, Coxeter diagrams) that determine the dihedral group D_6.

b) Determine which dihedral groups are rigid.

c) Prove that if Γ_S and $\Gamma_{S'}$ are two Coxeter diagrams that determine isomorphic groups and all edges of Γ_S are labelled ∞, then the same is true for all edges of $\Gamma_{S'}$.

Exercise 0.4. A Coxeter group given by (W, S) is called a right-angled Coxeter group (RACG) if $m_{st} \in \{2, \infty\}$ for all $s \neq t$ in S.

a) Describe the graph property (in terms of both Coxeter graph and presentation graph) that characterizes those special subgroups of RACGs that are spherical.

Geometric reflection groups

Exercise 0.5 (One dimensional geometric reflection groups in $\mathbb{E}^1$)**.** Consider the real line with the Euclidean metric $\mathbb{E}^1$. The only non-singleton convex polytopes are intervals. Let P be one such interval, say $[-1/2, 1/2]$. Let $\rho_1 : x \mapsto -x-1$ and $\rho_2 : x \mapsto -x+1$ be the reflections about the faces $\{-1/2\}$ and $\{1/2\}$ respectively.

a) Show that the group generated by ρ_1 and ρ_2 is isomorphic to the infinite dihedral group $\mathcal{D}_\infty$.

b) Show that all geometric reflection groups defined on $\mathbb{E}^1$ are conjugate in the group of affine transformations of $\mathbb{E}^1$.

Exercise 0.6 (One dimensional geometric reflection groups in $\mathbb{S}^1$)**.** Show that geometric reflection groups in $\mathbb{S}^1$ are exactly the finite dihedral groups. (Hint: Construct an action of $\mathcal{D}_n$ on $\mathbb{S}^1$ and show that it is faithful.)

Exercise 0.8 (Two dimensional geometric reflection groups in $\mathbb{S}^2$ and $\mathbb{E}^2$)**.** A Coxeter polygon is a Coxeter polytope in $\mathbb{S}^2, \mathbb{E}^2$ or $\mathbb{H}^2$ that is a fundamental domain for a geometric reflection group.

a) Show that the dihedral angle at each vertex of a Coxeter polygon is an even integral fraction of 2π.

Let $\mathbb{X}^2$ be $\mathbb{S}^2$ (resp. $\mathbb{E}^2$) with constant curvature = +1 (resp. 0). If P is a polygon in $\mathbb{X}^2$ with area $A(P)$, dihedral angles $\theta_i = \pi/m_i$ and Euler characteristic $\chi(P) = 1$, then the Gauss-Bonnet theorem says that

$$A(P)\kappa(\mathbb{X}^2) + \sum_{i=1}^{n}(\pi - \pi/m_i) = 2\pi\chi(P) = 2\pi.$$

b) Use the above formula to enumerate all possible Euclidean Coxeter polygons with the possible dihedral angles.

c) Use the above formula to enumerate all possible spherical Coxeter polygons with the possible dihedral angles.

d) *What can you say about hyperbolic Coxeter triangles using the above formula?

Exercise 0.9. A triangle group $\Delta(p,q,r)$ is a Coxeter group defined by the Coxeter matrix

$$\begin{pmatrix} 1 & p & r \\ p & 1 & q \\ r & q & 1 \end{pmatrix}$$

a) For $p,q,r < \infty$, when is $\Delta(p,q,r)$ a spherical/Euclidean/hyperbolic geometric reflection group?

b) Consider the action of $\Delta(2,2,\infty)$ on $\mathbb{E}^2$. What is the fundamental domain?

c) **Consider $\Delta(\infty,\infty,\infty)$ acting on $\mathbb{H}^2$. Draw the fundamental domain. Show that Δ has a finite index subgroup that is isomorphic to a free group. Show that Δ is not a geometric reflection group. Hint: Use the notion of quasi-isometry and quasi-isometry invariants.

CAT(K)

C2 Exercise: show that a tree is CAT(K) for any $K \in \mathbb{R}$.

C3 Exercise: If X is CAT(K) for $K < K'$ then X is also CAT(K')
(But not vice versa).

C4 Exercise: show that a graph with only finitely different edge lengths is CAT(1) $\Leftrightarrow$ it does not contain an isometrically embedded cycle of length $< 2\pi$

C5 Exercise: show that there is a unique geod between two points at dist $< D_K$ in a CAT(K) space.
(CAT(0) spaces are uniquely geodesic).

C6

Exercise: let X be a CAT(0) space.
show that local geodesics are geodesics.

$\gamma : [0,\infty) \to X$ a path is called a local geod
if $\forall t \geq 0$, $\exists \varepsilon$ st $\gamma|_{(t-\varepsilon, t+\varepsilon)}$ is a geodesic.

Hint: $G = \{ t \geq 0 \mid \text{st } \gamma|_{[0,t]} \text{ is a geod} \}$.
$G \neq \phi$ b/c γ is a geod in a nhd of 0.
show G closed
show G open

C7

Exercise: A map $\phi : X \to Y$ b/w two metric spaces is called a local isometry if $\forall x \in X$, $\exists$ nhd B_x of x s.t. $\phi|_{B_x}$ is an isometric embedding.

If $\phi : X \to Y$ is a local isometry from
X geod metric space to
Y a CAT(0) space
then show that ϕ is an isometric embedding.

Davis complex

Exercises:

D1 Show that K is a fundamental domain for the act^n of $W \curvearrowright \Sigma$

D2. Show that $W \curvearrowright \Sigma$ prop disc. and cocompactly

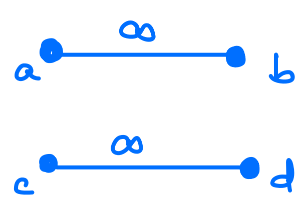
D3. show that Σ is simply connected.

D4. Suppose (W,S) is a Coxeter system such that every proper special subgroup is finite. Show that K is a simplex.

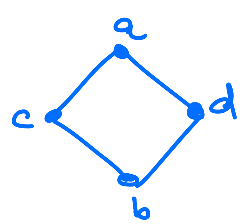
Exercise D5. RACG.

$W = D_\infty \times D_\infty$

$\Gamma =$

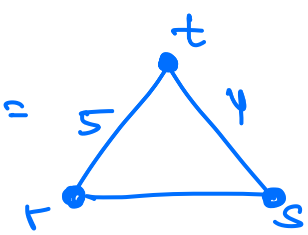


$\Gamma =$

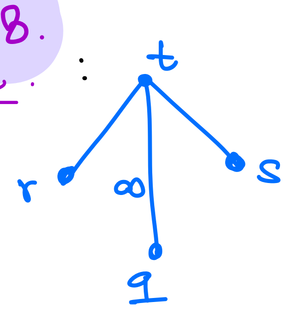


1. Draw K, L, Γ presentation graph.
2. Do you see a relationship between L and Γ.
2. Write $\mathcal{S} = \{T \subseteq S \mid T \text{ spherical}\}$.
3. Draw cells $\Sigma_T(d_T)$.
4. glue using L.

Exercise D6. $\Gamma =$



Exercise D8. :



Exercise D7